Estimation of the DOAs of Coherent Signals in Beam Space Processing for Phased Arrays

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***Abstract*—Standard high resolution DOA estimation techniques such as MUSIC fail when the received signals are correlated. However spatial smoothing of the element space (ES) covariance matrix of receiver outputs overcomes the rank deficiency of the signal subspace and allows high accuracy DOA estimates to be obtained. In beam space (BS) array processing the ES covariance matrix is not available and spatial smoothing techniques cannot be directly applied. In this paper an ES covariance matrix reconstituted from the BS covariance matrix is proposed to which spatial smoothing techniques can be applied.**

Keywords—Beam space; DOA estimation; correlated signals; spatial smoothing; phased arrays

I. INTRODUCTION

When estimating the Direction of Arrival (DOA) of multiple signals using phased arrays it is commonly assumed that the various signals are uncorrelated. For example, the phases of the returns from multiple raindrops are randomly distributed and with a large number of samples, the reflected signals from different directions can be treated as independent. However, in many applications, signals from different DOAs can be correlated. For example, multipath often happens, e.g., signals which scatterer off a target can be then reflected and bounced off adjacent objects such as roads, buildings and superstructures to produce correlated returns from different directions with different phase delays and amplitudes. When the same signal is transmitted in different directions, the returns from large, stationary targets at the same range can be coherent provided the scattering does not change over the coherent processing interval.

The problem of DOA estimation has received considerable attention in array processing where many techniques based on using the covariance matrix of phased array receiver outputs (termed ES: Element Space) have been developed. When signals from different directions are coherent, standard techniques such as MUSIC (Multiple Signal Classification) fail due to rank deficiencies of the signal covariance matrix and methods using the outputs from different subarrays to recover the rank loss of the "signal subspace" are commonly used. In these approaches, slight differences between the covariance matrices of different subarrays are exploited to address the rank deficiency problem. Termed spatial smoothing, techniques such as forward/backward averaging [1] or its extensions, e.g., [2], can be used to restore the rank of signal covariance matrix for uniform linear arrays (ULAs), by forming a new matrix whose elements are the averaged correlations of subarray receivers, and whose rank is the same as the number of signals. Then, eigen-space methods can be successfully applied to estimate the DOAs of coherent signals but at the expense of a lower angular resolution, as the aperture of each subarray is smaller than the original array.

In beam space (BS) array processing only the outputs of preformed beams are available and since its first proposal and application to real data [3,4] it has been shown that optimum beamforming techniques and high resolution DOA techniques such as MUSIC and its derivatives can be implemented using just the beam outputs when the incident signals are uncorrelated [5]. However for coherent signals existing BS approaches are not directly applicable, since the receiver subarrays necessary for spatial smoothing are not available and the Vandermonde structure of the ES steering vector is not present in the equivalent BS steering vectors. Thus the problem of using subspace methods to estimate the DOAs of correlated signals remains an open question for BS array processing. This paper proposes a new BS technique for estimating the DOA of coherent signals by reconstructing a synthesized ES covariance matrix the elements of which can then be used to form the subarrays necessary for spatial smoothing.

The BS algorithm considered in this paper considers only receive beams that are formed over a sub-sector of the whole angular space and thus the number of beams may be significantly smaller than the number of receivers. For the radar application considered here the implicit assumption has been made that the transmit system forms a wide beam that covers the whole sector of interest and so the transmit beam pattern can be ignored. Whilst this is true for some radar systems other systems use the same hardware to form both the transmit and receive beams and then sequentially scan a single TX/RX beam over the sector of interest. In this case the standard BS algorithm [3] is not applicable and a new approach, [6], has been developed which explicitly takes into account the transmit beam pattern. However this paper is restricted to standard BS algorithms where the radar's transmit pattern is assumed to be uniform over the sector of interest.

The rest of the paper is organised as follows. In Section II BS processing is defined and it is shown how standard DOA estimation techniques fail when the incident signals are coherent. In Section III the relationship of the BS signal subspace to the corresponding ES one is derived and a synthetic ES covariance matrix suitable for spatial smoothing and DOA estimation is proposed. An example of the application of this to a ULA is given in Section IV.

II. Beam Space Processing and Coherent Signals

A. Beam Space Processing

Denoting the receiver outputs of a K element phased array at the t-th pulse by the $K\times 1$ vector

$$\boldsymbol{x}(t) = [x_1(t) \quad x_2(t) \quad \dots \quad x_K(t)]^T,$$

and dropping the pulse dependence, the ES covariance matrix, $\boldsymbol{R}_x$, is given by

$$\boldsymbol{R}_x = \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}^H\}.$$

A vector $\boldsymbol{y}$ of M conventional beam outputs is given by

$$\boldsymbol{y} = [y(\theta_1) \quad y(\theta_2) \quad \dots \quad y(\theta_M)]^T = \boldsymbol{V}_b^H\boldsymbol{x},$$

where $y(\theta)$ is given by

$$y(\theta) = \boldsymbol{v}^H(\theta)\boldsymbol{x},$$

and $\boldsymbol{V}_b$ is a $K\times M$ matrix containing the steering vectors for each of the various M beams and is given by

$$\boldsymbol{V}_b \equiv \boldsymbol{V}_b(\Theta_b) = [\boldsymbol{v}(\theta_1) \quad \boldsymbol{v}(\theta_2) \quad \dots \quad \boldsymbol{v}(\theta_M)],$$

where $\Theta_b = \{\theta_1, \quad \theta_2, \quad \dots, \quad \theta_M\}$ denotes the set of steering directions of the preformed beams.

The covariance matrix of the beam outputs [3] is given by

$$\boldsymbol{R}_y = \mathrm{E}\{\boldsymbol{y}\boldsymbol{y}^H\} = \boldsymbol{V}_b^H\boldsymbol{R}_x\boldsymbol{V}_b.$$

The output power of the optimum BS beamformer is given by

$$p_{MVDR-BS}(\theta) = \left(\boldsymbol{v}^H(\theta)\boldsymbol{V}_b\boldsymbol{R}_y^{-1}\boldsymbol{V}_b^H\boldsymbol{v}(\theta)\right)^{-1}.$$

B. Signal Model

For L signals the ES covariance matrix is given by

$$\boldsymbol{R}_x = \boldsymbol{V}_s\boldsymbol{R}_s\boldsymbol{V}_s^H + \sigma_n^2\boldsymbol{I}, \tag{1}$$

where the $K \times L$ matrix $\boldsymbol{V}_s(\Theta_s)$ contains the steering vectors, $\boldsymbol{v}(\theta_{si})$, corresponding to the DOAs, θ_{si}, of all the scatterer signal returns and is given by

$$\boldsymbol{V}_s \equiv \boldsymbol{V}_s(\Theta_s) = [\boldsymbol{v}(\theta_{s1}) \quad \boldsymbol{v}(\theta_{s2}) \quad \dots \quad \boldsymbol{v}(\theta_{sL})],$$

and $\boldsymbol{\sigma}_n^2$ is the power of the spatially uncorrelated receiver noise. The covariance matrix of the L signals is given by

$$\boldsymbol{R}_s = E\{\boldsymbol{s}\boldsymbol{s}^H\} = \begin{bmatrix} \sigma_{s1}^2 & \rho_{12}\sigma_{s1}\sigma_{s2} & \cdots & \rho_{1L}\sigma_{s1}\sigma_{sL} \\ \rho_{12}^*\sigma_{s1}\sigma_{s2} & \sigma_{s2}^2 & \cdots & \rho_{2L}\sigma_{s2}\sigma_{sL} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1L}^*\sigma_{s1}\sigma_{sL} & \rho_{2L}^*\sigma_{s2}\sigma_{sL} & \cdots & \sigma_{sL}^2 \end{bmatrix},$$

where the correlation coefficient between two signals $s_i(t)$ and $s_j(t)$ whose return powers are given by σ_{si}^2 and σ_{sj}^2 is given by

$$\rho_{ij} = \frac{E\{s_i(t)s_j^*(t)\}}{\sigma_{si}\sigma_{sj}}.$$

The resulting BS covariance matrix is

$$\boldsymbol{R}_y = \boldsymbol{V}_b^H\boldsymbol{R}_x\boldsymbol{V}_b = \boldsymbol{H}\boldsymbol{R}_s\boldsymbol{H}^H + \boldsymbol{V}_b^H\boldsymbol{V}_b\sigma_n^2, \tag{2}$$

where

$$\boldsymbol{H} \equiv \boldsymbol{H}(\Theta_s) = \boldsymbol{V}_b^H(\Theta_b)\boldsymbol{V}_s(\Theta_s) \equiv \boldsymbol{V}_b^H\boldsymbol{V}_s.$$

C. Beam Space and Coherent Arrivals

The degradation of the optimum ES beamformer in the presence of coherent signals was analysed in [7] and only the BS case for two coherent arrivals is considered here.

The receiver output vector is given by

$$\boldsymbol{x}(t) = s_1(t)\boldsymbol{v}(\theta_{s1}) + s_2(t)\boldsymbol{v}(\theta_{s2}) + \boldsymbol{n}(t),$$

and thus the BS covariance matrix is given by

$$\boldsymbol{R}_y = [\boldsymbol{h}(\theta_{s1}) \quad \boldsymbol{h}(\theta_{s2})]\begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}^*\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}[\boldsymbol{h}(\theta_{s1}) \quad \boldsymbol{h}(\theta_{s2})]^H + \sigma_n^2\boldsymbol{V}_b^H\boldsymbol{V}_b,$$

where $\boldsymbol{h}(\theta)$, the BS "steering vector", is given by

$$\boldsymbol{h}(\theta) = \boldsymbol{V}_b^H\boldsymbol{v}(\theta).$$

When the SNR of $s_1(t)$ is very high, i.e., $\sigma_{s1}^2 \gg \sigma_n^2$, the output power of the optimum BS beamformer, after some manipulations, can be simply approximated as

$$p_{MVDR-BS}(\theta_{s1}) \approx \sigma_{s1}^2(1 - |\rho_{12}|^2),$$

indicating that the output power depends both on the signal power at θ_{s1} and the correlation coefficient between the two signals. Higher correlation causes more attenuation to the output power, and a weak but coherent interference may significantly reduce the output power of a strong signal.

An example of the BS beamformer output for two fully coherent signals, where $\rho_{12} = j$, i.e., the direct path and reflected signals are 90° out of phase, was considered. A ULA containing 64 elements spaced half of a wavelength apart and 20 Tx/Rx beams were formed covering the azimuthal region of 1° to 20° with 1° separation. The returns from two scatterers have SNRs of 10 dB and 6 dB at any single receiver and they were incident from azimuthal angles 8° and 15° respectively. The theoretical output power of the BS conventional and BS MVDR beamformers and BS MUSIC are plotted versus azimuthal angle in Figure 1. (*Aside*: The conventional BS beamformer takes the M preformed beams and interpolates.) In this example, the conventional BS beamformer is only slightly affected by the coherence. Considering the array aperture size, the two targets were reasonably well separated in azimuth, but the performance of the optimum BS beamformer and BS MUSIC were seriously degraded by the coherence.

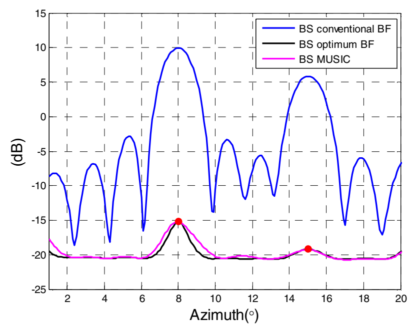

Fig. 1. *Outputs of the conventional, optimum BS beamformers and MUSIC for two coherent signals*

It was assumed that the number of signals was known, thus the dimension of the signal subspace was chosen as two. The BS MUSIC output plotted in Figure 2 was normalised for comparison but was actually 18 dB lower than shown.

III. ES and BS Subspaces

D. Beam Space Rank Deficiency

Subspace methods utilise the eigen-decomposition of the covariance matrix to separate the signal and noise subspaces. When the signals are uncorrelated or partially correlated, the signal covariance matrix is non-singular and its rank depends on the number of signals L. (It is assumed that L is less than both the number of beams and the number of receivers.) In this case the L steering vectors corresponding to the DOAs of signals span the signal subspace and are orthogonal to the noise subspace allowing the DOAs to be estimated.

However, when the signals are completely coherent and ignoring irrelevant propagation losses, the beam output vector can be written as

$$\boldsymbol{y}(t) = s_0(t)\boldsymbol{V}_b^H \sum_{l=1}^{L} \rho_l \sigma_{sl} \boldsymbol{v}(\theta_{sl}) + \boldsymbol{V}_b^H \boldsymbol{n}(t),$$

where $s_0(t)$ is the radar transmit signal with unit amplitude and

$$\rho_l = \frac{E\{s_l(t)s_0^*(t)\}}{\sigma_{sl}\sigma_{s0}}, (\sigma_{s0} = 1).$$

The BS covariance matrix is given by

$$\boldsymbol{R}_y = \boldsymbol{h}_s \boldsymbol{h}_s^H + \sigma_n^2 \boldsymbol{V}_b^H \boldsymbol{V}_b,$$

where

$$\boldsymbol{h}_s = \sum_{l=1}^{L} \rho_l \sigma_{sl} \boldsymbol{h}(\theta_{sl}) = \boldsymbol{V}_b^H \sum_{l=1}^{L} \rho_l \sigma_{sl} \boldsymbol{v}(\theta_{sl}) = \boldsymbol{V}_b^H \boldsymbol{v}_s.$$

The signal subspace, defined by $\boldsymbol{h}_s$, is of rank one and when the beams are orthogonal, i.e., $\boldsymbol{V}_b^H \boldsymbol{V}_b = \boldsymbol{I}$, $\boldsymbol{h}_s$ is a scalar times the principal eigenvector of the BS covariance matrix, $\boldsymbol{q}_{y1}$, i.e.,

$$\boldsymbol{q}_{y1} = \frac{\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|} e^{j\varphi},$$

where $e^{j\varphi}$ is an arbitrary phase shift. The remaining eigenvalues are $\lambda_2 = \lambda_3 = \cdots = \lambda_M = \sigma_n^2$, and their corresponding eigenvectors, $\boldsymbol{q}_{yi}$, $i = 2,3,\cdots,M$, span the BS noise subspace. (*NB*: beams normalised so that ${\boldsymbol{V}_b}^H \boldsymbol{V}_b = \boldsymbol{I}$ are used hereafter.)

Although $\boldsymbol{h}_s$ is a linear combination of the BS steering vectors of the coherent signals, it is not equal to any vector lying on the manifold of BS steering vectors. Consequently, subspace methods, such as MUSIC, fail in DOA estimation, as no sharp output peaks would appear in the BS array manifold to indicate the DOAs. This is the reason for the poor performance of BS MUSIC shown in Figure 1.

Aside: When the beams are not orthogonal the Sector Focussed Stability (SFS) method [4] can be used to generate beams that are mutually orthogonal, where the generalised SFS coherent BS steering vector is given by

$$\boldsymbol{h}'(\theta) = \left(\boldsymbol{V}_b^H \boldsymbol{V}_b\right)^{-\frac{1}{2}} \boldsymbol{h}(\theta).$$

E. ES and BS Subspace Relationships

Inspection of (1) and (2) suggests the following relationships between the ES and BS subspaces

a. Under the transformation $\boldsymbol{V}_b$ the BS noise eigenvectors are a subset of the ES noise eigenvectors.

b. The ES signal subspace can be recovered from the BS signal subspace via the same transformation

Whilst this may seem intuitive the formal proofs below offer an interesting insight into the issue of the number of beams required in BS processing.

First the eigen-decompositions of the ES and BS covariance matrices are summarised:

In ES

$$\boldsymbol{R}_x = \boldsymbol{V}_s \boldsymbol{R}_s \boldsymbol{V}_s^H + \sigma_n^2 \boldsymbol{I} = \boldsymbol{Q}_s \boldsymbol{\Lambda}_s \boldsymbol{Q}_s^H + \sigma_n^2 \boldsymbol{I}$$

In BS

$$\boldsymbol{R}_y = \boldsymbol{V}_b^H \boldsymbol{V}_s \boldsymbol{R}_s \boldsymbol{V}_s^H \boldsymbol{V}_b + \sigma_n^2 \boldsymbol{I} = \boldsymbol{Q}_{ys} \boldsymbol{\Lambda}_{ys} \boldsymbol{Q}_{ys}^H + \sigma_n^2 \boldsymbol{I}$$

where orthonormality of the preformed beams, $\boldsymbol{V}_b^H \boldsymbol{V}_b = \boldsymbol{I}$, has been again assumed. Since the columns of $\boldsymbol{V}_s$ and $\boldsymbol{Q}_s$ both span the same signal subspace the orthonormality of the ES signal and noise subspaces is given by

$$\boldsymbol{V}_s^H \boldsymbol{Q}_n = \boldsymbol{Q}_s^H \boldsymbol{Q}_n = \boldsymbol{0}.$$

Since $\boldsymbol{V}_b$ is of full rank and $M > L$ the dimensions of the signal subspaces in both ES and BS are identical, i.e., $L = L'$. Similarly for BS the columns of $\boldsymbol{V}_b^H \boldsymbol{V}_s$ and $\boldsymbol{Q}_{ys}$ span the same signal subspace and so

$$\boldsymbol{V}_s^H \boldsymbol{V}_b \boldsymbol{Q}_{yn} = \boldsymbol{Q}_{ys}^H \boldsymbol{Q}_{yn} = \boldsymbol{0}.$$

Thus

$$\boldsymbol{R}_x \boldsymbol{V}_b \boldsymbol{Q}_{yn} = \boldsymbol{V}_s \boldsymbol{R}_s \boldsymbol{V}_s^H \boldsymbol{V}_b \boldsymbol{Q}_{yn} + \sigma_n^2 \boldsymbol{V}_b \boldsymbol{Q}_{yn} = \boldsymbol{\sigma}_n^2 \boldsymbol{V}_b \boldsymbol{Q}_{yn} \quad (3)$$

Thus proving assertion (a) that a subset (*M-L*) of the ES noise eigenvectors can be recovered from the BS noise eigenvectors through multiplication by the matrix $\boldsymbol{V}_b$.

Aside: Note that in proving the above assertion no assumption has been made about the rank of $\boldsymbol{R}_s$, i.e., this result is valid for both coherent and uncorrelated arrivals.

To address the relationship of the ES and BS signal subspaces attention is restricted to the case of coherent signals.

In the Appendix it is shown that when $\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2} = 1$,

$$\boldsymbol{q}_i^H \boldsymbol{V}_b \boldsymbol{q}_{y1} = 0, i \geq 2,$$

thus demonstrating that $\boldsymbol{V}_b \boldsymbol{q}_{y1}$ is orthogonal to the ES noise subspace and coherent with the ES generalised coherent signal steering vector. Indeed, as the following proof shows

$$\boldsymbol{v}_s = \frac{\|\boldsymbol{v}_s\|^2}{\|\boldsymbol{h}_s\|^2} \boldsymbol{V}_b \boldsymbol{h}_s = \frac{\|\boldsymbol{v}_s\|^2}{\|\boldsymbol{h}_s\|} \boldsymbol{V}_b \boldsymbol{q}_{y1} e^{-j\varphi}.$$

Proof: Using (3) the eigen-decomposition of $\boldsymbol{R}_x$ is given by

$$\boldsymbol{R}_x = \frac{\sigma_s^2 + \sigma_n^2}{\|\boldsymbol{v}_s\|^2} \boldsymbol{v}_s \boldsymbol{v}_s^H + \sum_{i=2}^{M} \sigma_n^2 \boldsymbol{V}_b \boldsymbol{q}_{yi} \boldsymbol{q}_{yi}^H {\boldsymbol{V}_b}^H + \sum_{i=M+1}^{K} \sigma_n^2 \boldsymbol{q}_i \boldsymbol{q}_i^H,$$

where σ_s^2 is the combined power of the coherent signals and is given by $\sigma_s^2 = \|\boldsymbol{v}_s\|^2$.

From the orthogonality of the beams and the BS eigenvectors it follows that

$$\boldsymbol{R}_x \frac{\boldsymbol{V}_b \boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2} = \frac{\sigma_s^2 + \sigma_n^2}{\|\boldsymbol{v}_s\|^2} \boldsymbol{v}_s + \sum_{i=M+1}^{K} \sigma_n^2 \boldsymbol{q}_i \boldsymbol{q}_i^H \boldsymbol{V}_b \boldsymbol{V}_b^{\boldsymbol{H}} \frac{\boldsymbol{v}_s}{\|\boldsymbol{h}_s\|^2},$$

and the Appendix shows that for $M < i \leq K$, $\boldsymbol{q}_i^H \boldsymbol{V}_b \boldsymbol{V}_b^{\boldsymbol{H}} = 0$ thus giving

$$\boldsymbol{R}_x \frac{\boldsymbol{V}_b \boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2} = \frac{\sigma_s^2 + \sigma_n^2}{\|\boldsymbol{v}_s\|^2} \boldsymbol{v}_s,$$

and the uniqueness of the principal eigenvector provides the required result.

To summarise

a. The ES signal subspace can be reconstituted from the BS signal subspace using $\tilde{\boldsymbol{v}}_s = \boldsymbol{V}_b \boldsymbol{q}_{y1}$.

b. *M-1* ES noise eigenvectors can be reconstituted from the BS noise eigenvectors using $\boldsymbol{V}_b \boldsymbol{q}_{yi}$ for $2 < i \leqslant M$.

c. When the beams are not orthogonal the transformation is of the form of the Moore Penrose pseudo inverse of the BS transformation, i.e.,

$$\boldsymbol{V}_b \left(\boldsymbol{V}_b^{\boldsymbol{H}} \boldsymbol{V}_b\right)^{-\boldsymbol{1}} \boldsymbol{q}_{yi},$$

Discussion

The above results are valid only if the value of $\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2}$ equals one and an investigation was carried out exploring the variation of the above ratio as a function of the number of beams used. Consider two coherent signals at DOAs of 8.23° and 9.44° (two random but closely spaced angles) with SNRs of 10 dB and 5 dB respectively, the correlation coefficient was e^j, and beams were chosen to be equally evenly separated in angle covering a sector of interest from 1° to 20°. The value of $\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2}$ versus the number of beams is shown in Figure 2. It indicates the value of $\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2}$ converges to one when the number of beams is larger than 12, in which case the spacing between the beam centres is less than half a beamwidth. Whilst a formal proof of this result has not been possible, it appears to hold in general for different SNR levels and DOAs.

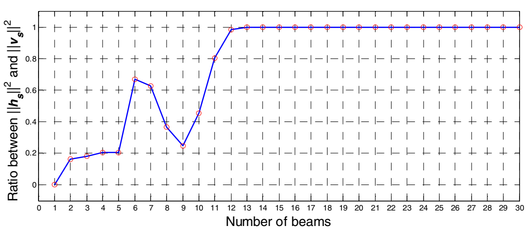

Fig. 2. *The value of* $\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2}$ *versus the number of beams in a sector of interest*

F. ES Covariance Matrix Reconstruction

From the above summary we first consider reconstructing the ES covariance matrix, $\widetilde{\boldsymbol{R}}_x$, as

$$\widetilde{\boldsymbol{R}}_x = (\boldsymbol{V}_b^{\boldsymbol{H}})^+ \boldsymbol{R}_y \boldsymbol{V}_b{}^+,$$

where $^+$ denotes the Moore-Penrose pseudoinverse. When $M > K$ and the rows of $\boldsymbol{V}_b$ are independent, its pseudo inverse is given by

$$\boldsymbol{V}_b{}^+ = \boldsymbol{V}_b^{\boldsymbol{H}} (\boldsymbol{V}_b \boldsymbol{V}_b{}^H)^{-1},$$

and substituting in the above expression for $\widetilde{\boldsymbol{R}}_x$ gives

$$\widetilde{\boldsymbol{R}}_x = (\boldsymbol{V}_b \boldsymbol{V}_b^{\boldsymbol{H}})^{-1} \boldsymbol{V}_b \boldsymbol{V}_b^{\boldsymbol{H}} \boldsymbol{R}_x \boldsymbol{V}_b \boldsymbol{V}_b^{\boldsymbol{H}} (\boldsymbol{V}_b \boldsymbol{V}_b^{\boldsymbol{H}})^{-1} = \boldsymbol{R}_x.$$

When the number of independent beams is equal to the number of receivers, the ES covariance matrix can be exactly recovered using the expression $\widetilde{\boldsymbol{R}}_x = (\boldsymbol{V}_b^{\boldsymbol{H}})^{-1} \boldsymbol{R}_y \boldsymbol{V}_b{}^{-\boldsymbol{1}} = \boldsymbol{R}_x$. Thus the exact ES covariance matrix can be reconstructed when the number of beams, *M*, is greater than or equal to the number of receivers, *K*, and there are *K* independent beams. However, as most commonly happens in practice, the main interest is the case where $M < K$ independent beams are used and in this case the pseudoinverse matrix of $\boldsymbol{V}_b$ is given by

$$\boldsymbol{V}_b{}^+ = (\boldsymbol{V}_b^{\boldsymbol{H}} \boldsymbol{V}_b)^{-1} \boldsymbol{V}_b{}^H.$$

In this case, the higher dimensional ES covariance matrix cannot be perfectly reconstructed by applying a linear transformation to the lower dimensional BS covariance matrix. However as shown above, for coherent arrivals, the ES signal subspace can be recovered from the BS signal subspace by a matrix transformation and a subset (*M-1*) of the ES noise eigenvectors can be recovered from the BS noise eigenvectors. Thus the eigen-spectrum of $\widetilde{\boldsymbol{R}}_x$ will have one dominant value, $\sigma_s^2 + \sigma_n^2$, corresponding to $\boldsymbol{v}_s$, *M-1* values of σ_n^2 corresponding to the noise eigenvectors and *K-M* zero eigenvalues, reflecting the lack of full rank of the reconstituted ES covariance matrix.

This reconstituted ES matrix could be then used as input to the standard spatial smoothing algorithm but for DOA estimation an alternative approach is proposed based on one assumption and one property of the MUSIC DOA estimation algorithm. The assumption is that the noise output of receiver elements is spatially white, i.e., uncorrelated between receivers and so the reconstructed ES noise covariance matrix can be assumed to be a $K \times K$ diagonal matrix. The used property is that for MUSIC the only way noise power affects the algorithm is in setting the threshold for determining the dimension of the signal subspace, i.e., the number of signals. Thus for DOA estimation using the subspace method, the exact signal and noise power values $\tilde{\sigma}_s^2$ and $\tilde{\sigma}_n^2$ are not required, as they will not directly affect the result of DOA estimation.

Thus for mutually orthogonal beams a reconstituted ES covariance matrix is defined as

$$\widetilde{\boldsymbol{R}}_x' = \tilde{\sigma}_s^2 \boldsymbol{V}_b \boldsymbol{q}_{y1} \boldsymbol{q}_{y1}^H \boldsymbol{V}_b^{\boldsymbol{H}} + \tilde{\sigma}_n^2 \boldsymbol{I}, \quad (4)$$

where $\tilde{\sigma}_s^2$ and $\tilde{\sigma}_n^2$ are arbitrary, but in order to show the DOAs as very sharp peaks $\tilde{\sigma}_s^2 > \tilde{\sigma}_n^2$ is used.

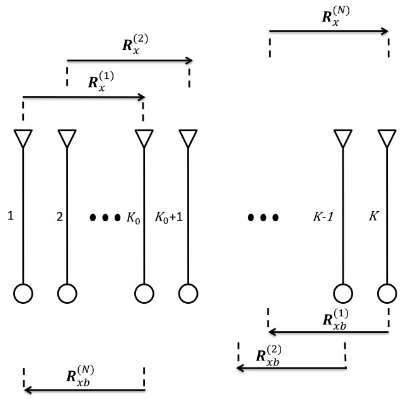

Fig. 3 *The forward/backward spatial smoothing scheme.* K_0 *is the number of the elements in each subarray.*

G. Spatial Smoothing using Reconstructed ES

The spatial smoothing used is the standard backward/forward approach [1] whereby subarrays are formed by starting at the opposite ends of the array. This is illustrated in Figure 3 and since the details are well documented [1,2,7] the mathematical expressions are not repeated here.

IV. NUMERICAL EXAMPLES

The scenario used to produce Figure 2 was used again and the BS covariance matrix was modelled as (2). After applying the SFS transformation, the reconstructed covariance matrix is given by (4) where $\tilde{\sigma}_s = 1$ and $\tilde{\sigma}_n = 0.1$. Then, forward spatial smoothing was applied to $\widetilde{\boldsymbol{R}}'_x$ with one average and ES MUSIC algorithm was then applied. The outputs of MUSIC using different numbers of beams are shown in Figure 4 where the true DOAs are plotted as magenta spots. It shows that DOA estimation is very difficult using a small number of beams whose separation is larger than half the 3 dB beamwidth. The MUSIC output peaks appear at the correct DOAs, with the peaks becoming sharper as the number of beams is increased.

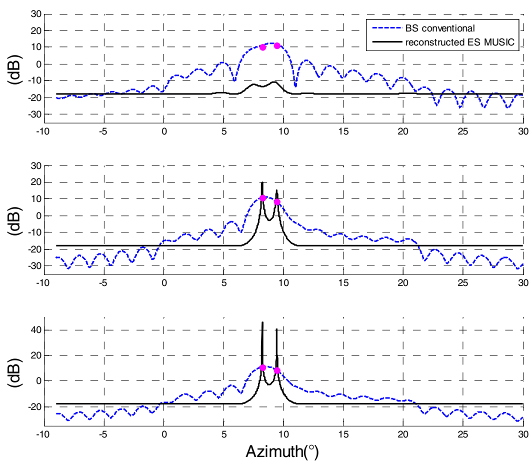

Fig. 4 *An example of applying spatial smoothing and MUSIC algorithms to a reconstructed ES covariance matrix of a model containing two coherent signals. The numbers of beams are 10, 14 and 20 respectively.*

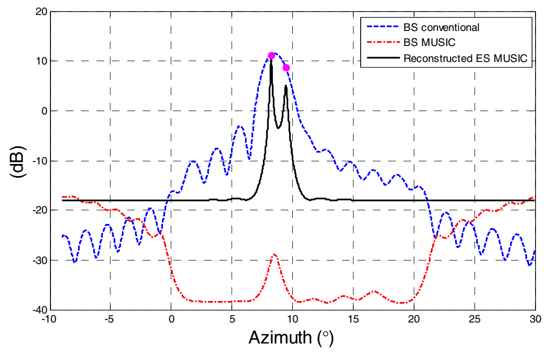

Fig. 5 *A simulation example of applying spatial smoothing and MUSIC algorithms to the reconstructed ES covariance matrix of two coherent signals.*

The same scenario with 20 beams at 1° separation was simulated and 1024 independent samples were generated for each beam. After reconstructing the ES covariance matrix $\widetilde{\boldsymbol{R}}'_x$ and applying forward spatial smoothing with one average, the output of MUSIC versus azimuthal angle is plotted in Figure 5. Similarly to the theoretical numerical analysis, the DOAs of coherent signals can be estimated accurately, but the output peaks are not so sharp. The output of the BS conventional beamformer and BS MUSIC are also plotted for comparison. Neither the conventional BS beamformer nor the BS MUSIC approaches can separate the two DOAs and the output values of the latter one are very small.

V. DISCUSSION

Provided a sufficient number of beams are used to cover a given sector of interest an ES covariance matrix can be reconstructed from the estimated BS covariance matrix. This then permits a limitation of standard BS processing, i.e., the inability to estimate the DOAs of coherent signals, to be overcome. The use of the Moore Penrose pseudo inverse allows the ES signal subspace to be exactly recovered and hence standard spatial smoothing algorithms can be applied. Further use of the pseudo inverse generates a subset of the ES noise subspace eigenvectors. Numerical and simulated examples have been presented to verify the proposed algorithm.

It should be noted in passing that the reconstruction procedure proposed here is not limited to the situation whereby the incident signals are coherent – the proposed approach can readily be applied to the case where the incident signals are uncorrelated although the motivation for reconstructing an ES space covariance matrix may not be as strong.

APPENDIX

The eigen-decomposition of $\boldsymbol{R}_y$ is given by

$$\boldsymbol{R}_y = \sum_{i=1}^{M} \lambda_{yi} \boldsymbol{q}_{yi} \boldsymbol{q}_{yi}^H,$$

where

$$\lambda_{y1} = \frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2}\sigma_s^2 + \sigma_n^2 > \sigma_n^2, \lambda_{yi} = \sigma_n^2, (i \geq 2),$$

and

$$\boldsymbol{q}_{y1} = \frac{\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|} e^{j\varphi},$$

and the steering vectors have been normalized so that $\boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}$ when the beams are orthogonal and since no confusion occurs the subscript *b* has been dropped.

Since eigenvectors are mutually orthogonal it follows that

$$\boldsymbol{q}_{yi}^H \boldsymbol{h}_s = 0, \ \ 2 \leqslant i \leqslant M.$$

From the eigen-decomposition of $\boldsymbol{R}_y$ it follows that

$$\boldsymbol{V}\boldsymbol{R}_y\boldsymbol{V}^H = \left(\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2}\sigma_s^2 + \sigma_n^2\right)\boldsymbol{V}\boldsymbol{q}_{y1}\boldsymbol{q}_{y1}^H\boldsymbol{V}^H + \sigma_n^2\sum_{i=2}^{M}\boldsymbol{V}\boldsymbol{q}_{yi}\boldsymbol{q}_{yi}^H\boldsymbol{V}^H. \quad \text{(A.1)}$$

From the eigen-decomposition of $\boldsymbol{R}_x$, i.e.,

$$\boldsymbol{R}_x = (\sigma_s^2 + \sigma_n^2)\boldsymbol{q}_1\boldsymbol{q}_1^H + \sigma_n^2\sum_{i=2}^{K}\boldsymbol{q}_i\boldsymbol{q}_i^H,$$

it follows that

$$\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{R}_x\boldsymbol{V}\boldsymbol{V}^H = \frac{\sigma_s^2 + \sigma_n^2}{\|\boldsymbol{v}_s\|^2}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{v}_s\boldsymbol{v}_s^H\boldsymbol{V}\boldsymbol{V}^H + \sigma_n^2\sum_{i=2}^{M}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H + \sigma_n^2\sum_{i=M+1}^{K}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H,$$

where the first *M-1* noise eigenvectors are chosen to be the ones that satisfy $\boldsymbol{q}_i = \boldsymbol{V}\boldsymbol{q}_{yi}$ and so the RHS of the above equation can be rewritten as

$$\frac{\sigma_s^2+\sigma_n^2}{\|\boldsymbol{v}_s\|^2}\boldsymbol{V}\boldsymbol{h}_s\boldsymbol{h}_s^H\boldsymbol{V}^H + \sigma_n^2\sum_{i=2}^{M}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{V}\boldsymbol{q}_{yi}\boldsymbol{q}_{yi}^H\boldsymbol{V}^H\boldsymbol{V}\boldsymbol{V}^H + \sigma_n^2\sum_{i=M+1}^{K}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H,$$

which for independent beams reduces to

$$\frac{\|\boldsymbol{h}_s\|^2(\sigma_s^2 + \sigma_n^2)}{\|\boldsymbol{v}_s\|^2}\boldsymbol{V}\boldsymbol{q}_{y1}\boldsymbol{q}_{y1}^H\boldsymbol{V}^H + \sigma_n^2\sum_{i=2}^{M}\boldsymbol{V}\boldsymbol{q}_{yi}\boldsymbol{q}_{yi}^H\boldsymbol{V}^H + \sigma_n^2\sum_{i=M+1}^{K}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H. \quad \text{(A.2)}$$

Equating (A.1) and (A.2) gives

$$\left(\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2}\sigma_s^2 + \sigma_n^2\right)\boldsymbol{V}\boldsymbol{q}_{y1}\boldsymbol{q}_{y1}^H\boldsymbol{V}^H = \frac{\|\boldsymbol{h}_s\|^2(\sigma_s^2 + \sigma_n^2)}{\|\boldsymbol{v}_s\|^2}\boldsymbol{V}\boldsymbol{q}_{y1}\boldsymbol{q}_{y1}^H\boldsymbol{V}^H + \sigma_n^2\sum_{i=M+1}^{K}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H.$$

When $\frac{\|\boldsymbol{h}_s\|^2}{\|\boldsymbol{v}_s\|^2} = 1$ it follows that

$$\sigma_n^2\sum_{i=M+1}^{K}\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H = 0.$$

Since $\sigma_n^2 > 0$ and $\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H \geq 0$ then

$$\boldsymbol{q}_{M+1}^H\boldsymbol{V}\boldsymbol{V}^H = \boldsymbol{q}_{M+2}^H\boldsymbol{V}\boldsymbol{V}^H = \cdots \boldsymbol{q}_K^H\boldsymbol{V}\boldsymbol{V}^H = 0, (M < i \leq K),$$

and so

$$\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{q}_{y1} = \boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H\boldsymbol{V}\boldsymbol{q}_{y1} = 0, \ \ M < i \leq K. \quad \text{(A.3)}$$

Thus

$$\boldsymbol{R}_x\frac{\boldsymbol{V}\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2} = \left(\frac{\sigma_s^2 + \sigma_n^2}{\|\boldsymbol{v}_s\|^2}\boldsymbol{v}_s\boldsymbol{v}_s^H + \sum_{i=2}^{M}\sigma_n^2\boldsymbol{q}_i\boldsymbol{q}_i^H + \sum_{i=M+1}^{K}\sigma_n^2\boldsymbol{q}_i\boldsymbol{q}_i^H\right)\frac{\boldsymbol{V}\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2}$$

$$= \frac{\sigma_s^2 + \sigma_n^2}{\|\boldsymbol{v}_s\|^2}\boldsymbol{v}_s\frac{\boldsymbol{v}_s^H\boldsymbol{V}\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2} + \sum_{i=2}^{M}\sigma_n^2\boldsymbol{V}\boldsymbol{q}_{yi}\boldsymbol{q}_{yi}^H\frac{\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2} + \sum_{i=M+1}^{K}\sigma_n^2\boldsymbol{q}_i\boldsymbol{q}_i^H\boldsymbol{V}\boldsymbol{V}^H\frac{\boldsymbol{v}_s}{\|\boldsymbol{h}_s\|^2}.$$

where orthogonality of the beams, i.e., $\boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}$ has been assumed. From the orthogonality of the beam space eigenvectors the second term is zero and using (A.3) the third term is zero. Thus

$$\boldsymbol{R}_x\frac{\boldsymbol{V}\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2} = \frac{\sigma_s^2+\sigma_n^2}{\|\boldsymbol{v}_s\|^2}\boldsymbol{v}_s\frac{\boldsymbol{h}_s^H\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2}.$$

Since $(\sigma_s^2 + \sigma_n^2)\boldsymbol{v}_s = \boldsymbol{R}_x\boldsymbol{v}_s$, the above can be rewritten as

$$\|\boldsymbol{v}_s\|^2\boldsymbol{R}_x\frac{\boldsymbol{V}\boldsymbol{h}_s}{\|\boldsymbol{h}_s\|^2} = \boldsymbol{R}_x\boldsymbol{v}_s,$$

and because $\boldsymbol{R}_x$ is a full rank and invertible matrix, hence

$$\boldsymbol{v}_s = \frac{\|\boldsymbol{v}_s\|^2}{\|\boldsymbol{h}_s\|^2}\boldsymbol{V}\boldsymbol{h}_s.$$

Since $\boldsymbol{h}_s = e^{-j\varphi}\|\boldsymbol{h}_s\|\boldsymbol{q}_{y1}$,

$$\boldsymbol{v}_s = \frac{\|\boldsymbol{v}_s\|^2}{\|\boldsymbol{h}_s\|}\boldsymbol{V}\boldsymbol{q}_{y1}e^{-j\varphi}.$$

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