Fast Direction-of-Arrival Estimation in Coprime Arrays

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***Abstract*—We present a fast direction-of-arrival (DOA) estimation algorithm for coprime array structures. In many practical applications of radar signals processing, reducing the redundancy inherent in linear arrays is advantageous. Arrays employing a coprime-pair configuration offer an increased number of degrees of freedom and permit the detection of more sources than sensors. Traditionally, this is achieved using conventional subspace-based methods. While these methods, such as MUltiple Signal Classification (MUSIC), are capable of high-resolution, the computational complexity of these algorithms is a significant hindrance to their practical implementation as the number of sensors increases. Furthermore, they must resort to spatial smoothing in order to calculate the covariance matrix. We, on the other hand, propose a fast iterative interpolated beamforming algorithm that has a computational complexity of the same order as the fast Fourier transform (FFT). The proposed approach is capable of delivering high fidelity DOA estimates that outperform the traditional high resolution methods. Simulation results demonstrate the DOA estimation performance of the technique.**

***Index Terms*—FIIB, DOA estimation, coprime arrays, SNR, MUSIC**

I. Introduction

The commercial use of radar has seen significant advances in the design and application of array signal processing techniques for direction of arrival (DOA) estimation [1]. With the emergence of radar behind technologies such as automotive sensing and health devices, the problem of multiple-source DOA estimation arises across a broad range of radar applications [2]–[5]. As such, the ability of increasing the degrees of freedom (DOFs) of the sensing array has been the subject of widespread research. Thus, coprime arrays and coprime time samplers have received significant attention due to their ability to provide large apertures and increased degrees of freedom (DOFs) with low cost [6]. In the context of automotive radar, coprime configurations allow for more sources than there are sensors to be detected [7]. This is done by recovering all the lags of an extended array with more sensors that the sum of the two coprime arrays [6]. The resulting vector of lags is then treated as a single snapshot of the extended array, allowing DOA estimation to be carried out.

In order to capitalize on the promise of coprime arrays, conventional DOA estimation approaches rely on subspace-based techniques such as MUSIC, Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) and their derivatives [8], [9]. These so-called high-resolution methods can resolve up to $N-1$ sources with an N element array [10]. However, these algorithms require high signal-to-noise (SNR) ratios and multiple snapshots to provide accurate performance. In many cases, operating conditions and physical constraints may restrict the available number of snapshots with which to perform reliable DOA estimation. Furthermore, the computational complexity of these subspace-based techniques is a significant disadvantage in applications that require the rapid sensing of a large number of targets. Thus, much of the literature has focused on overcoming these hurdles and optimizing their performance using techniques such as spatial smoothing [11], the development of Compressed Sensing (CS) theory and alternative matrix completion methods [12]–[14].

In this paper, we propose the Fast Iterative Interpolated Beamformer (FIIB) to coprime arrays in order to address, in one swipe, both the computational complexity and single snapshot problem. The proposed algorithm employs the Fast Fourier Transform (FFT) to locate the peaks corresponding to the far-field sources [15]. Then, through an iterative interpolation search of the spatial spectrum coefficients, refined source DOA estimates are obtained. We show that the application of the FIIB in conjunction with a coprime configuration enables reliable and accurate detection of a higher number of sources than there are array elements. By virtue of its construction, the FIIB is a single snapshot algorithm and is therefore, ideal for application to the virtual signal vector in the coprime case. Moreover, the use of the FFT means that the FIIB is computationally fast. In fact, it outperforms subspace-based methods such as Root-MUSIC algorithm on both counts of accuracy, and efficiency of implementation.

The remainder of this paper is organized as follows. In Section II, we derive the signal model and coprime array preliminaries. Section III describes the proposed FIIB algorithm and its implementation to the coprime array problem. Section IV presents the results and discussion of the simulatory scenario, followed by Section V, which concludes this paper.

II. Problem Formulation

We start by reviewing coprime arrays and the popular subspace-based estimation method Root-MUSIC.

A. Coprime Arrays

A coprime array structure is shown in Figure 1, where M and N describe a coprime pair, with $M < N$. The unit inter-element spacing d is set to $\lambda/2$, where λ denotes the wavelength. As the first element is shared by both sub-arrays, the total number of elements constituting the array is then $K = M + N - 1$.

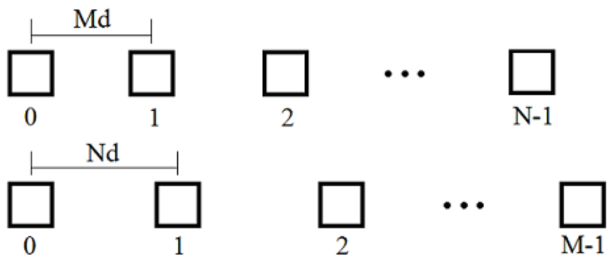

Fig. 1: The conventional coprime array configuration [16]. The first element in both sub-arrays is shared.

The positions of the array elements are described by

$$\mathrm{P} = \{Mnd|\ 0 \leq n \leq N-1\} \cup \{Nmd|\ 0 \leq m \leq M-1\}. \quad (1)$$

We may denote $\mathbf{p} = \left[p_1, \ldots, p_K\right]^{\mathrm{T}}$ as the vector describing the positions of the sensors constituting the array, where $p_i \in \mathrm{P}$ for $i = 1, \ldots, K$ and the first sensor is assumed to be the reference sensor, i.e. $p_1 = 0$. Consider Q uncorrelated signals, with baseband waveforms with $s_q(t)$, where $t = 1, \ldots, T$ for $q = 1, \ldots, Q$, impinging on the array from angles $\Theta = \left[\theta_1, \ldots, \theta_Q\right]^{\mathrm{T}}$. The time-domain data vector received by the coprime array is then expressed as

$$\mathbf{x}(t) = \sum_{q=1}^{Q} \mathbf{a}(\theta_q)\mathrm{s}_q(t) + \mathbf{n}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \qquad (2)$$

where $\mathbf{a}(\theta_q) = \left[1, e^{j\frac{2\pi p_2}{\lambda}\sin(\theta_q)}, \ldots, e^{j\frac{2\pi p_K}{\lambda}\sin(\theta_q)}\right]$ is the steering vector of the coprime array corresponding to the signal coming from direction θ_q and $\mathbf{s}(t) = \left[s_1(t), \ldots, s_Q(t)\right]^{\mathrm{T}}$. The noise vector $\mathbf{n}(t)$ is taken to be zero-mean Gaussian with covariance matrix $\sigma^2\mathbf{I}_K$.

The covariance matrix obtained from the time-domain data vector $\mathbf{x}(t)$ is then easily shown to be given by

$$\mathbf{R}_{\mathrm{xx}} = \mathbf{E}\left[\mathbf{x}(t)\mathbf{x}^{\mathrm{H}}(t)\right] = \mathbf{A}\mathbf{R}_{\mathrm{ss}}\mathbf{A}^{\mathrm{H}} + \sigma^2\mathbf{I}_K, \qquad (3)$$

where $\mathbf{R}_{\mathrm{ss}} = \mathbf{E}\left[\mathbf{s}(t)\mathbf{s}^{\mathrm{H}}(t)\right] = \mathrm{diag}(\left[\sigma_1^2, \ldots, \sigma_Q^2\right])$ is the covariance matrix of the source, with σ_q^2 symbolising the input signal power of the qth source.

Although the covariance matrix contains holes, its coprime structure enables an increased number of DOFs to be achieved. By vectorizing $\mathbf{R}_{\mathrm{xx}}$, we have

$$\mathbf{w} = \mathrm{vec}(\mathbf{R}_{\mathrm{xx}}). \qquad (4)$$

The elements of the virtual signal vector in (4) may be viewed as resulting from a virtual array with sensors having positions given by the union of the cross difference set and self difference sets pertaining to the physical array. These virtual positions are described respectively by

$$L_c = \{\pm(Mn - Nm)d\}, 0 \leq n \leq N-1, 0 \leq m \leq M-1 \quad (5)$$

and

$$L_s = \{l_s | l_s = Mn\} \cup \{l_s | l_s = Nm\}. \qquad (6)$$

The number of unique integers in the union set of L_c and L_s directly determines the number of distinct cross-correlation lags in the covariance matrix, which then gives the number of available DOFs. By strategically selecting part or all of the virtual sensors to perform DOA estimation with, the number of sources able to be detected by the array is increased. However, the existence of ‘holes’ in the cross difference set L_c, which is due to missing cross-correlation lags, complicates the DOA performance [10]. One strategy to rectify this is to double the number of array elements within the smaller subarray. The coprime pair then become N and $2M$, where the constraint of $M < N$ still holds. As the zeroth sensor position of the two subarrays are collocated, the total number of physical elements constituting the array then becomes $K = 2M + N - 1$, with the minimum inter-element spacing remaining as $\lambda/2$. It was shown in [17] that this modification to the configuration gives $2MN + 1$ degrees of freedom as opposed to physical array, which provides only $2M + N - 1$ degrees of freedom.

The extended coprime configuration of $2M$ and N elements will be adopted for the rest of the paper. Following from (4), we then define $\mathbf{y}$ as the vector containing those elements within $\mathbf{w}$ corresponding to the lags that lie within the range of $-MN$ to MN. The DOA estimation algorithms are then applied to the new vector $\mathbf{y}$.

B. Root-MUSIC Algorithm

It was shown, e.g. in [6], that the virtual signal $\mathbf{y}$ has a structure that is similar to that of $\mathbf{x}$ above, which then permits subspace-based methods to be applied to it. In this section we consider the Root-MUSIC algorithm [18].

The well-known MUSIC algorithm carries out a subspace decomposition of the covariance matrix of the vector $\mathbf{y}$. Since $\mathbf{y}$ acts like a single snapshot, spatial smoothing is generally employed. In the coprime case, we use the maximum smoothing window length of MN. Therefore, the estimate of the covariance matrix of $\mathbf{y}$ becomes

$$\mathbf{R}_{\mathrm{yy}} = \frac{1}{MN+1} \sum_{l=1}^{MN+1} \mathbf{y}_l \mathbf{y}_l^{\mathrm{H}}, \qquad (7)$$

where $\mathbf{y}_l = [y_l, \ldots, y_{l+MN}]^T$. Once the covariance matrix, $\mathbf{R}_{yy}$, is obtained, it is decomposed into signal and noise subspaces. Let $\mathbf{V}$ be the $MN \times (MN - Q)$ matrix comprising the eigenvectors of the noise subspace. Since the steering vectors of the Q sources are orthogonal to $\mathbf{V}$, the MUSIC algorithm finds the source DOAs by locating the peaks of the MUISC spectrum

$$\mathrm{P}_{\mathrm{MUSIC}}(\phi) = \frac{1}{\mathbf{b}^{\mathrm{H}}\mathbf{V}\mathbf{V}^{\mathrm{H}}\mathbf{b}}, \qquad (8)$$

where $\mathbf{b}$ is an arbitrary steering vector of length MN.

The evaluation of the MUSIC spectrum is carried out on some appropriately chosen grid, with the accuracy of the DOA estimates being limited by the density of the grid. An alternative approach, adopted by Root-MUSIC, uses the fact that the steering vector b can be expressed as

$$\begin{aligned}\mathbf{b}(\theta) &= \left[1, e^{j\frac{2\pi \mathbf{y}(1)}{\lambda}\sin(\theta)}, \ldots, e^{j\frac{2\pi y(MN-1)}{\lambda}\sin(\theta)}\right] \\ &= \left[1, z, \ldots, z^{MN-1}\right] \\ &= \mathbf{b}(z).\end{aligned}$$

Then the denominator of the MUSIC spectrum can be recast as the polynomial

$$Q(z) = \mathbf{b}^{\mathrm{H}}(z)\mathbf{V}\mathbf{V}^{\mathrm{H}}\mathbf{b}(z), \tag{9}$$

with the roots $z_q = e^{j\frac{2\pi}{\lambda}\mathbf{y}\sin(\theta_q)}$ corresponding to the true source DOA. giving the DOAs. Then, the true DOA can be determined from those roots that are close to the unit circle. Once the closest roots are identified, the DOA are given as

$$\theta_q = \sin^{-1}\left(\frac{\lambda}{2\pi d}\arg(z_q)\right). \tag{10}$$

III. Proposed Fast Iterative Interpolated Beamforming Estimator

In the multiple source case, Fourier-based interpolation methods garner less appeal due to their inherent biases induced by spectral leakage [19], hence the popularity of the subspace-based methods. The proposed FIIB technique is a non-parametric algorithm based upon the FFT that is able to achieve unbiased and accurate estimation of multiple source DOAs. The FIIB employs an estimate-and-subtract strategy so as to extract the sources successively in an inner loop. The results are then passed to an outer loop which refines these estimates, eliminating the bias at convergence. At the algorithm's core resides a simple yet highly accurate interpolation strategy that is combined with a leakage subtraction scheme. This combination endows the algorithm with excellent convergence properties and the capability of tracking the Cramèr-Rao bound (CRB) at all SNRs, making it an effective approach for Fourier-based estimation in the multi-source case. Thus, the FIIB overcomes the limitations of the well-known Fourier beamforming (FB) [15] without resorting to the computationally expensive subspace decomposition that is used by traditional high resolution techniques.

A detailed description of the general implementation of the FIIB estimator is provided in Algorithm 1. For clarity of presentation, we let M denote the length of the input vector $\mathbf{y}$. The conventional beamforming coefficients $\mathbf{Y}[n]$ are obtained using the K-length FFT, where $K = rM$ for $r = 1$ or 2. All subsequent processing is carried out in the frequency domain, avoiding any further application of the FFT. During the first iteration, we obtain coarse estimates of the Q DOAs sequentially, beginning with the strongest source. Specifically, for the ℓ-th source, the previously estimated sources are subtracted from the signal, as shown in line 8 of Algorithm 1, with the highest peak of the spectrum being located in line 9. In this manner, the previously estimated $\ell - 1$ sources are removed, exposing the ℓ-th source [15]. Note that the terms $\mathbf{S}_i[n]$ are the DFT coefficients of the i-th source steering vector. The coarse estimate of the ℓ-th source is then refined using interpolation on the leakage-free Fourier coefficients in lines 12-15. Finally, the source amplitude is estimated in line 16.

Algorithm 1: Single Snapshot FIIB for DOA estimation

Input : The received data vector $\mathbf{y}$

1 Put $\hat{\nu}_q = 0$ and $\hat{\beta}_q = 0$ for $q = 1, \ldots, Q$
2 Let $\mathbf{Y} = \mathtt{FFT}(\mathbf{y}, K)$ with $r = 1$ or 2 and $K = rM$
3 Set $l = 0$
4 **for** *L iterations or convergence* **do**
5     $l = l + 1$
6   **for** $q = 1 : Q$ **do**
7   **if** $l = 1$ **then**
8             $\hat{\mathbf{Y}}[n] = \mathbf{Y}[n] - \sum_{\substack{i=1\\i\neq q}}^{Q} \hat{\beta}_i \mathbf{S}_i[n]$
9             $\hat{\nu}_q = \frac{1}{K}\max_{1\leq n\leq K}\left|\hat{\mathbf{Y}}[n]\right|^2$
10   **end**
11         $\hat{\mathbf{Y}}_{\pm p}(\hat{\nu}_q) = \mathbf{Y}_{\pm p}(\hat{\nu}_q) - \sum_{\substack{i=1\\i\neq q}}^{L} \hat{\beta}_i \mathbf{S}_i(\hat{\nu} \pm p),\ p = \frac{r}{2}$
12         $\hat{\mathbf{Y}}_p(\hat{\nu}_q) := \hat{\mathbf{Y}}(\hat{\nu}_q + p)$
13         $h = \mathtt{Re}\left[\frac{\hat{\mathbf{Y}}_p(\hat{\nu}_q) + \hat{\mathbf{Y}}_{-p}(\hat{\nu}_q)}{\hat{\mathbf{Y}}_p(\hat{\nu}_q) - \hat{\mathbf{Y}}_{-p}(\hat{\nu}_q)}\right]$
14         $\hat{\nu}_q \leftarrow \hat{\nu}_q + \frac{h}{2K}$
15         $\hat{\nu}_q \leftarrow \frac{\hat{\nu}_q}{K}$
16         $\hat{\beta}_q = \frac{1}{M}\left\{\sum_{k=0}^{M-1}\mathbf{x}[k]e^{-j\frac{2\pi}{K}k\hat{\nu}_q} - \sum_{\substack{i=1\\i\neq q}}^{Q}\hat{\beta}_i\hat{\mathbf{S}}_i(\hat{\nu}_q)\right\}$
17   **end**
18 **end**

Output: $\left[\hat{\beta}, \hat{\nu}_q\right]$

In the context of coprime arrays, the FIIB is applied to the virtual vector $\mathbf{y}$, which is of length $K = 2MN + 1$. The principal stages of the algorithm are summarized in the following sequence for each source:

1) Subtract all other sources,
2) Determine coarse estimate in first iteration,
3) Refine the DOA estimate,
4) Obtain the complex amplitude estimate ,
5) Calculate the DOA of source q as $\theta_q = \sin^{-1}(\nu_q/d)$.

The computational complexity of a DOA estimation method is a very important practical consideration. Despite their ability to resolve multiple signals, subspace-based algorithms are afflicted by high computational complexity. Root-MUSIC is known to have a computational complexity of $\mathrm{O}(M^3)$ [20].

The estimation performance and convergence of the FIIB algorithm are detailed in [21], where it was shown that the algorithm yielded unbiased estimates that are on the Cramér Rao Bound (CRB). The algorithm was demonstrated to converge at a rate determined by the relative magnitudes of the leakage and noise. It was shown that for low SNR values, the algorithm typically converged within a few iterations (as predicted by the theory) and therefore, incurs a low computational cost. The use of the FFT only during the first iteration endows the algorithm with a computational complexity of $\mathrm{O}(M \log_2 M)$. During each subsequent iteration the DOA estimates are updated by calculating only two new DFT coefficients, which requires a complexity of $\mathrm{O}(M)$. Thus, the operations employed by the loop possess only an order of M and the complexity of the algorithm remains of order $M \log_2 M$. It is then evident that it has a significantly lower computational cost than the Root-MUSIC algorithm.

IV. Simulation Results

We consider the following scenarios in which we compare the performance of the proposed FIIB algorithm with Root-MUSIC: detecting more sources than there are sensors, determining the RMSE, high-resolution detection capabilities and the computational complexity. In all scenarios, we employ 1000 independent runs and model the noise as zero-mean Gaussian with variance $\sigma_n^2 = 1$. The zero-padding amount for the FIIB was set to the default value of 2, while the number of iterations used was 40.

A. Detecting more sources than sensors

In this example, we consider a coprime array configuration with $M = 5$ and $N = 7$, giving a total of 16 sensors. This produces a virtual array with 71 unique contiguous lags. We consider $Q = 25$ sources distributed between -60^{o} and 60^{o}, which is more than the available number of physical sensors. We set the source amplitudes to 1 and the SNR to 0 dB. The sources are distributed uniformly about the azimuth in the range of -60^{o} to 60^{o}. The covariance matrix, $\mathbf{R}_{XX}$ is estimated using 500 snapshots.

Figure 2 depicts the spectrum of the noiseless signal. Also shown are the true DOAs as well as the estimates obtained from both the FIIB and Root-MUSIC averaged over the 1000 simulation runs. It is clear that both algorithms are able to successfully detect all 25 sources. Also we see that the estimator achieves super-resolution performance since some of the sources are closer together than a beamwidth, where a beamwidth is defined as $\frac{1}{MN+1}$ in frequency. In fact, the three sources at each end are closer together than the beamwidth.

B. RMSE of FIIB vs. Root-MUSIC

Now in order to evaluate the error performance, we consider a scenario involving two sources. The RMSE of the angle estimates yielded by FIIB and Root-MUSIC were obtained over the SNR values ranging from -30 to 100 dB. In each run, the source DOAs, θ_1 and θ_2, were given by $\theta_1 = \theta_0 - \Delta\theta/2$ and $\theta_2 = \theta_0 + \Delta\theta/2$, where θ_0 was randomly generated over the interval [-30,30] and the source separation $\Delta\theta$ was set to 30^{o}. It can be seen from Figure 3 that the estimation

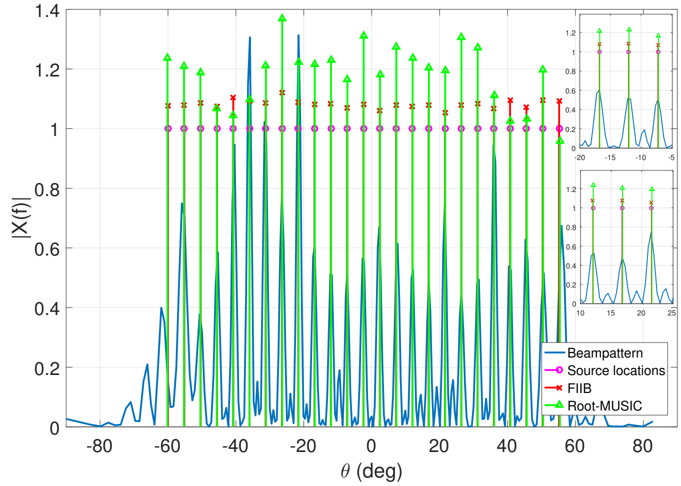

Fig. 2: Spectrum for FIIB and Root-MUSIC for the coprime pair $M = 5$ and $N = 7$, with $Q = 25$ sources embedded in SNR $= 0$ dB noise.

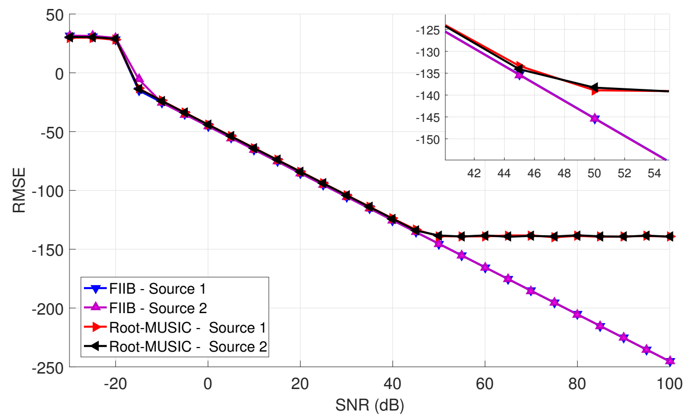

Fig. 3: RMSE of FIIB and Root-MUSIC for the coprime pair $M = 5$ and $N = 7$ for two sources randomly distributed along the azimuth.

performance of FIIB is marginally better than that of Root-MUSIC for SNR values -30 dB to +45 dB. For SNR values above +45 dB, however, we see that Root-MUSIC tapers off whereas FIIB continues its linear trend.

C. Super-resolution capability

In this example, we again employ the coprime configuration with $M = 5$ and $N = 7$. To test the high-resolution capabilities, five sources were distributed in clusters of close proximity along the azimuth, with $\theta_q = [-55, -52, 5, 58, 62]$. We again embedded the sources within noise at SNR $= 0$ dB.

It can be seen from Figure 4 that FIIB and the Root-MUSIC are capable of resolving the sources when in close proximity with one another. It is clear that although it is a FFT-based estimator, the FIIB is capable of resolving sources that are not otherwise resolved by the conventional beamformer. Therefore, the FIIB exhibits super-resolution performance, rivaling that of Root-MUSIC.

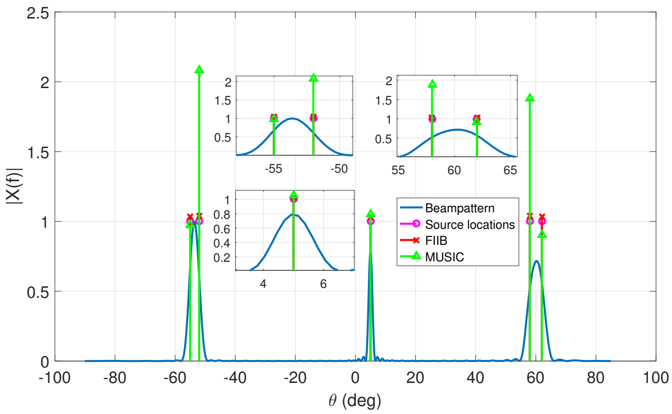

Fig. 4: Spectrum for FIIB and Root-MUSIC for the coprime pair $M = 5$ and $N = 7$, with sources located at -55^o, -52^o, 5^o, 58^o and 62^o for $\text{SNR} = 0$ dB.

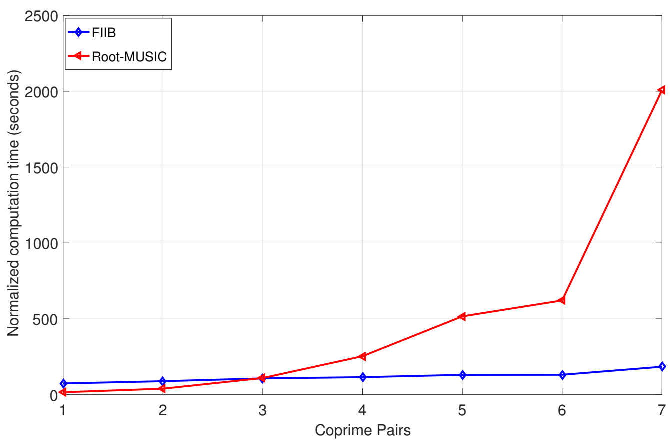

Fig. 5: Comparison of computational complexity for FIIB and Root-MUSIC for the different coprime pairs shown in Table I.

D. Computational complexity of FIIB vs. Root-MUSIC

Finally, the computational complexity of FIIB and Root-MUSIC were compared across the different coprime pairs shown in Table I. The same $Q = 25$ sources were uniformly distributed from -60^o to 60^o and embedded in noise at $\text{SNR} = 0$ dB.

The execution time of the FIIB and Root-MUSIC algorithms for each coprime pair were recorded and averaged over the simulation runs. In order to account for the specific processor used, the execution times for each algorithm were then normalized to that of the FFT.

Figure 5 shows that the computational complexity of Root-MUSIC far exceeds that of the FIIB. For coprime pairs 1 and 2, the computational cost of Root-MUSIC appears lower than that of the FIIB since the former used an optimized MATLAB implementation. However, as the size of the array becomes larger, the cost of the subspace decomposition becomes dominant and Root-MUSIC becomes far more computationally expensive than FIIB. Notice that, despite the increase in coprime array sizes, the complexity of the FIIB remains flat, demonstrating that it is of the same order as the FFT.

Coprime Pair	M	N
1	5	7
2	7	9
3	9	11
4	11	13
5	12	17
6	13	17
7	17	19

TABLE I: Different coprime configurations used to benchmark the computational complexity of FIIB and Root-MUSIC.

V. Conclusion

In this paper, we examined the application of the FIIB algorithm to the coprime array structure and compared its performance to the popular Root-MUSIC subspace-based technique. We proposed a fast and accurate estimation algorithm that outperforms high-resolution based algorithms such as Root-MUSIC. We showed that The new FIIB approach enjoys a vastly lower computational complexity than the more taxing Root-MUSIC algorithm. Furthermore, we demonstrated through simulation results that FIIB is significantly better than Root-MUSIC in terms of estimation accuracy.

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