# Generalized Design Approach for Fourth-order Difference Co-array


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***Abstract*—This paper presents a generalized approach to design fourth-order non-uniform linear arrays (NLAs) by combining any two second-order NLAs (e.g. co-prime array (CA), nested array (NA)). This new-formed NLA could exactly achieve a large hole-free range in its fourth-order difference co-array. By establishing the relationship between the second-order NLAs and the fourth-order NLAs, closed-form expressions for the physical sensor locations and the corresponding virtual sensor configurations are specifically derived. Compared to the prior proposed structure, the new structure could effectively increase the virtual aperture, therefore, having the ability to detect more sources. By considering direction of arrival (DOA) estimation for quasi-stationary (QS) signals, numerical simulation results validate the effectiveness of the proposed array structure.**



***Keywords*—*DOA estimation; non-uniform linear arrays; fourth-order co-array; difference co-array of the sum co-array; quasi-stationary signals***


## I. Introduction

For the traditional direction of arrival (DOA) estimation, uniform linear arrays (ULAs) combining with high-resolution sub-space fitting algorithms (e.g. MUSIC [1], ESPRIT [2] and their variants) are deeply investigated and widely used. However, one limitation is that the maximum number of resolvable sources and the physical aperture are only $O(\mathrm{N})$ for the ULA with $N$ sensors. Therefore, to increase the degrees of freedom (DOFs), non-uniform linear arrays (NLAs) have been attracted much attentions, including MRA [3], (generalized) co-prime array (CA) [4]–[7] and (super/augmented) nested array (NA) [8]–[10]. These sparse array structures could generate the second-order difference co-array (SODC) and achieve a compressive sampling for the second-order covariance matrix [11], [12], which could be further used to conduct DOA estimation. As a result, DOFs are increased from $O(\mathrm{N})$ to $O(\mathrm{N}^2)$.

Recently, to further extend the virtual array, the fourth-order difference co-array (FODC) has been proposed by calculating the difference co-array of the prior SODC. Compared to the second-order covariance matrix, this new formed fourth-order statistics could further increase the DOFs to $O\left(\mathrm{N}^4\right)$. Furthermore, two well-known methods are proposed for FODC construction. For the first method, the FODC is constructed by using the cumulant-based algorithm [13]. The application of $2q$th-order cumulants is presented in [14]–[16] as well. The limitation of such cumulant-based methods is that they can not be applied on Gaussian sources. To solve this problem, the wide-sense quasi-stationary (QS) signals structure [17]–[19] is exploited to form the FODC. Accordingly, two kinds of sparse array geometries, namely fourth-order co-prime array (FOCA) [18] and fourth-order nested array (FONA) [19], are proposed by adding an extra sub-array to the conventional CA and NA. These two fourth-order NLAs are proved to be with high performances, but it is noted that how to design an associated sub-array for other NLAs, such as super/augmented NA, is still a blank.

In this paper, we present a generalized NLA design method, where for any given second-order NLA included but not limited to the CA and the NA (denoted as parent NLA), one could design an extra sub-array (denoted as partner NLA). Specifically, this partner sub-array is also designed based on one second-order NLA, but its inter-element spacing is multiplied by a scale factor. The fourth-order NLA of interest is made up of a parent NLA and a partner NLA. By establishing the relationship between the second-order NLAs and the fourth-order NLAs, closed-form expressions for the physical sensor locations and the corresponding virtual sensor configurations are specifically derived. It shows that such generalized NLA could generally contain a large hole-free range and keep a low mutual coupling effect. If chosen CA or NA as parent NLA, respectively, our proposed structure has a larger consecutive range of FODC and could resolve more number of sources compared with the prior FOCA and FONA in [18], [19].

## II. Review of the Fourth-order Difference Co-array

In order to describe the virtual array configurations, we start with introducing a few set operations [20].

**Definition 1** (**Set Operation**). *For any two given sets of integers $\mathbb{A}$, $\mathbb{B}$ and any integer $\beta$ belonging to the set of all integers $\mathbb{Z}$, one can define the following operations:*

$$\begin{aligned}
\textbf{\textit{Union Set:}}&\quad \mathbb{A}\cup\mathbb{B}=\{a\mid\forall a\in\mathbb{A}\ or\ a\in\mathbb{B}\}\\
\textbf{\textit{Sum Set:}}&\quad \mathbb{A}+\mathbb{B}=\{a+b\mid\forall a\in\mathbb{A},\forall b\in\mathbb{B}\}\\
\textbf{\textit{Difference Set:}}&\quad \mathbb{A}-\mathbb{B}=\{a-b\mid\forall a\in\mathbb{A},\forall b\in\mathbb{B}\}\\
\textbf{\textit{Self-difference Set:}}&\quad \mathbb{A}-\mathbb{A}=\{a-b\mid\forall a\in\mathbb{A},\forall b\in\mathbb{A}\}\\
\textbf{\textit{Dilation Set:}}&\quad \beta\mathbb{A}=\{\beta a\mid\forall a\in\mathbb{A},\beta\in\mathbb{Z}\}
\end{aligned}$$

Different from the stationary non-Gaussian signals which are usually used in the cumulant-based methods, the wide-sense QS signals we exploit contain several segments of uncorrelated stationary frames, which is more common in daily life. There are two basic assumptions concerning the property of QS signals [18].

***Assumption 1:*** With $K$ narrowband far-field wide-sense QS signals impinging on the array from directions $\theta_k$, where $k = 1, 2, \cdots, K$, the input sources $s_k(p,q)$ are wide-sense QS signals within the frame length $Q$, where $p = 1, 2, \cdots, P$ is the frame index and $P$ is the total number of frames. Then, the power of the $k$th signal in the $p$th frame $\delta^2_{p,k} = \mathrm{E}\{s_k(p,q)s^*_k(p,q)\}$ for $q \in \{(p-1)\cdot Q, (p-1)\cdot Q + 1, \cdots, p\cdot Q - 1\}$ can be approximated by

$$\delta^2_{p,k} \approx \frac{1}{Q}\sum_{q=(p-1)\cdot Q}^{p\cdot Q-1} s_k(p,q)s^*_k(p,q). \tag{1}$$

***Assumption 2:*** $\delta^2_{p,k}$ are wide-sense stationary and uncorrelated with each other. Hence, we have

$$\begin{aligned} m_k = \mathrm{E}\left\{\delta^2_{p,k}\right\}, \quad \hat{\delta^2_k} = \mathrm{E}\left\{(\delta^2_{p,k} - m_k)^2\right\}, \\ \mathrm{E}\left\{\left(\delta^2_{p,k_1} - m_{k_1}\right)\cdot\left(\delta^2_{p,k_2} - m_{k_2}\right)\right\} = 0, k_1 \neq k_2. \end{aligned} \tag{2}$$

Consider a linear array consisting of $N$ sensors at location $\mathbb{S} = \{p_1, p_2, \cdots, p_N\}$, where $p_n$ is the position of $n$th sensor, $n = 1, 2, \cdots, N$. It is noted that all the element positions are in units of half a minimal wavelength. Hence the $q$th received snapshot of the $p$th frame can be expressed as

$$\mathbf{x}(p,q) = \mathbf{A}\mathbf{s}(p,q) + \mathbf{w}(p,q), \tag{3}$$

where $\mathbf{s}(p,q)$ and $\mathbf{w}(p,q)$ denote the input source vector and the noise vector, respectively; $\mathbf{A}$ denotes an $N \times K$-dimensional array manifold matrix, whose $(n\text{th}, k\text{th})$ element $\mathbf{A}_{n,k}$ is $e^{-j\pi p_n \sin(\theta_k)}$, where $n = 1, 2\cdots, N$, $k = 1, 2, \cdots, K$.

Under ***Assumption 1***, by collecting $Q_p$ snapshots, the covariance matrix of the $p$th frame can be obtained by

$$\begin{aligned} \mathbf{R_{xx}}(p) =& \mathrm{E}\left\{\mathbf{x}(p,q)\mathbf{x}(p,q)^H\right\} = \mathbf{A}\mathbf{R_{ss}}(p)\mathbf{A}^H + \sigma^2\mathbf{I} \\ \approx& \frac{1}{Q}\sum_{(q=p-1)\cdot Q}^{p\cdot Q-1} \mathbf{x}(p,q)\mathbf{x}(p,q)^H, \end{aligned} \tag{4}$$

where $\mathbf{R_{ss}}(p) = diag\{[\delta^2_{p,1}, \delta^2_{p,2}, \cdots, \delta^2_{p,K}]\}$ denotes the source covariance matrix, $\sigma^2$ represents the noise power, and $\mathbf{I}$ is the $N \times N$ identity matrix.

$\mathbf{R_{xx}}(p)$ can be vectorized into a long vector $\mathbf{z}(p)$ :

$$\mathbf{z}(p) = vec\left\{\mathbf{R_{xx}}(p)\right\} = \underbrace{(\mathbf{A}^* \odot \mathbf{A})}_{\mathbf{A}_{\text{DC}}}\tilde{\mathbf{s}}(p) + \sigma_n^2\mathbf{l}_N, \tag{5}$$

where the term $(\mathbf{A}^* \odot \mathbf{A})$ is the manifold matrix of a difference co-array whose sensor positions are given by $\bar{\mathbb{C}} = \mathbb{S} - \mathbb{S}$ and $\odot$ represents the Khatri-Rao product. $\tilde{\mathbf{s}}(p) = [\delta^2_{p,1}, \delta^2_{p,2}, \cdots, \delta^2_{p,K}]^T$ is the virtual input source. $\mathbf{l}_N$ is a $N^2\times 1$ vector obtained by vectorizing $\mathbf{I}$.

With ***Assumption 2***, the expectation of $\mathbf{z}(p)$ can be calculated by

$$\begin{aligned} \bar{\mathbf{z}} = \mathrm{E}\left\{\mathbf{z}(p)\right\} =& \mathbf{A}_{\text{DC}}\mathrm{E}\left\{\tilde{\mathbf{s}}(p)\right\} + \sigma^2\mathbf{l}_N \\ =& \mathbf{A}_{\text{DC}}\bar{\mathbf{s}} + \sigma^2\mathbf{l}_N, \end{aligned} \tag{6}$$

where $\bar{\mathbf{s}}$ is the expectation of $\tilde{\mathbf{s}}(p)$.

In order to construct a zero-mean process, subtracting $\bar{\mathbf{z}}$ from $\mathbf{z}(p)$, we obtain

$$\tilde{\mathbf{z}}(p) = \mathbf{z}(p) - \bar{\mathbf{z}} = \mathbf{A}_{\text{DC}}(\tilde{\mathbf{s}}(p) - \bar{\mathbf{s}}). \tag{7}$$

Then, by collecting $P$ frames, the covariance matrix of virtual co-array signals $\tilde{\mathbf{z}}(p)$ can be obtained by $\mathbf{R_{zz}} = \mathrm{E}\left\{\tilde{\mathbf{z}}(p)\tilde{\mathbf{z}}(p)^H\right\}$. Similar with (5), $\mathbf{R_{zz}}$ can also be vectorized into a long vector

$$\mathbf{y} = vec\left\{\mathbf{R_{zz}}\right\} = (\mathbf{A}^*_{\text{DC}} \odot \mathbf{A}_{\text{DC}})\hat{\mathbf{s}}, \tag{8}$$

where $\hat{s}$ is $[\hat{\delta^2_1}, \hat{\delta^2_2}, \cdots, \hat{\delta^2_K}]^T$ and the term $(\mathbf{A}^*_{\text{DC}} \odot \mathbf{A}_{\text{DC}})$ is the manifold matrix of the difference co-array of SODC set $\bar{\mathbb{C}}$, whose sensor positions are given by

$$\tilde{\mathbb{C}} = \bar{\mathbb{C}} - \bar{\mathbb{C}}. \tag{9}$$

Here, the virtual array defined by $\tilde{\mathbb{C}}$ is often called fourth-order difference co-array (FODC). For sampling purpose, one needs to appropriately design a NLA $\mathbb{S}$ in order to maximize the hole-free range in $\tilde{\mathbb{C}}$. There are two array structures that have been proposed. One is the fourth-order co-prime array (FOCA) [18]. The other is the fourth-order nested array (FONA) [19]. The basic construction behind these two NLAs is adding an extra sub-array to the CA and the NA. Therefore, these two fourth-order arrays can be described as

$$\begin{aligned} \mathbb{S}_{\text{FOCA}} = \mathbb{S}_{\text{CA}} \cup \mathbb{S}_{\text{EXT\_CA}}, \\ \mathbb{S}_{\text{FONA}} = \mathbb{S}_{\text{NA}} \cup \mathbb{S}_{\text{EXT\_NA}}, \end{aligned} \tag{10}$$

where $\mathbb{S}_{\text{CA}}$ and $\mathbb{S}_{\text{NA}}$ are CA and NA, respectively; $\mathbb{S}_{\text{EXT\_CA}}$ and $\mathbb{S}_{\text{EXT\_NA}}$ denote the extra sub-arrays.

## III. Generalized Fourth-order Difference Co-array (G-FODC) Design

### A. G-FODC

In this paper, we focus on presenting a generalized design method. For any given NLA $\mathbb{S}_1$, such as MRA [3], (generalized) CA [4]–[6] and (super/augmented) NA [8]–[10], we can design an extra sub-array $\mathbb{S}_{\text{EXT}}$ to maximize the uniform one-side DOF. For the sake of brevity, the given NLA and the new designed sub-array are called parent NLA and partner NLA in this paper, respectively.

**Theorem 1** **(G-FODC).** *For any given parent NLA $\mathbb{S}_1$, assume that its uniform one-side DOF is $L_{u1}$. One can design its partner NLA as $\mathbb{S}_{EXT} = \lambda\mathbb{S}_2$, where $\lambda = 2L_{u1} + 1$, and $\mathbb{S}_2$ is any NLA with uniform one-side DOF as $L_{u2}$. Then, for the new formed NLA, i.e. $\mathbb{S}_1 \cup \mathbb{S}_{EXT}$, the uniform one-side DOF of its FODC is no less than $L_{u1} + \lambda L_{u2}$.*

*Proof:* Let $\mathbb{S} \triangleq \mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}}$. Then, based on the definition of the sum set and the difference set, one can directly obtain :

$$\tilde{\mathbb{C}} = (\mathbb{S} - \mathbb{S}) - (\mathbb{S} - \mathbb{S}) = (\mathbb{S} + \mathbb{S}) - (\mathbb{S} + \mathbb{S}). \tag{11}$$

It is indicated that the FODC behaves like a difference co-array of its sum co-array. Firstly, we will analyze the sum co-array of $\mathbb{S}$. Based on the property of the union set, we have $\mathbb{S}_1 \subset (\mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}})$ and $\mathbb{S}_{\text{EXT}} \subset (\mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}})$, then one can obtain :

$$\mathbb{S} + \mathbb{S} = (\mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}}) + (\mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}}) \supseteqq \mathbb{S}_1 + \mathbb{S}_{\text{EXT}}. \tag{12}$$

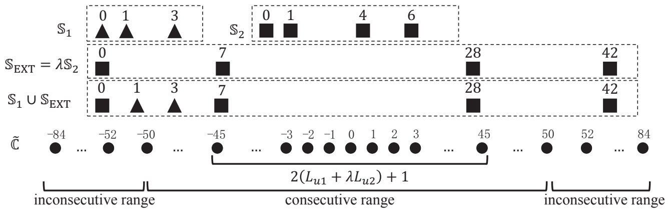


Fig. 1: Example of the proposed G-FODC array, where $\mathbb{S}_1$ and $\mathbb{S}_{\text{EXT}}$ denote the parent NLA and the partner NLA, respectively.

TABLE I: Summary of the G-FODC algorithm

| | |
|---|---|
| **Input:** | Two NLAs |
| **Output:** | Configurations of G-FODC |
| **Step 1:** | Consider two arbitrary NLAs set $\mathbb{S}_1$ and set $\mathbb{S}_2$, whose uniform one-side DOF are $L_{u1}$ and $L_{u2}$, respectively. |
| **Step 2:** | Design the new set : $\mathbb{S} = \mathbb{S}_1 \cup \lambda\mathbb{S}_2 = \mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}}$, where $\lambda = 2L_{u1} + 1$. |
| **Step 3:** | The corresponding SODC : $\bar{\mathbb{C}} = \mathbb{S} - \mathbb{S} = (\mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}}) - (\mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}})$. |
| **Step 4:** | The corresponding G-FODC : $\tilde{\mathbb{C}} = \bar{\mathbb{C}} - \bar{\mathbb{C}}$. |
| **Step 5:** | The uniform one-side DOF of $\tilde{\mathbb{C}}$ is no less than $L_{u1} + \lambda L_{u2}$. |

Combining (11) and (12), one can further conclude that

$$\tilde{\mathbb{C}} \supseteqq \underbrace{(\mathbb{S}_1 + \mathbb{S}_{\text{EXT}}) - (\mathbb{S}_1 + \mathbb{S}_{\text{EXT}})}_{\mathbb{D}_1}. \tag{13}$$

Therefore, the uniform one-side DOF of $\tilde{\mathbb{C}}$ is no less than that of $\mathbb{D}_1$.

To get the uniform one-side DOF results of $\mathbb{D}_1$, we will introduct *Theorem 1* in [20], where the authors have proved that, for aforementioned settings, the positive consecutive range (i.e. the uniform one-side DOF) of the difference co-array of $(\mathbb{S}_1 + \mathbb{S}_{\text{EXT}})$ is $L_{u1} + \lambda L_{u2}$. Therefore, one can conclude that the uniform one-side DOF of $\tilde{\mathbb{C}}$ must be no less than $L_{u1} + \lambda L_{u2}$. ■

An example of the new structre is shown in Figure 1, where the parent array $\mathbb{S}_1$ and the sub-array $\mathbb{S}_2$ are both MRA. As the uniform one-side DOF of $\mathbb{S}_1$ is $L_{u1} = 3$, the partner array $\mathbb{S}_{\text{EXT}} = \lambda\mathbb{S}_2 = \{0, 7, 28, 42\}$, where the scale factor $\lambda$ = 7. Therefore, the new set $\mathbb{S}_1 \cup \mathbb{S}_{\text{EXT}}$ is $\{0, 1, 3, 7, 28, 42\}$. The uniform one-side DOF of $\tilde{\mathbb{C}}$ is 50, which is larger than the $L_{u1} + \lambda L_{u2} = 45$. In addition, a MATLAB code for generating the proposed G-FODC can be found in [21].

The steps of designing G-FODC are summarized in Table I. We make the following remarks on **Theorem** 1 :

1. Interestingly, the number of DOFs in the proposed G-FODC has positive correlation with the largest consecutiveness of the parent array and the partner array in their SODC. While among all the existing sparse arrays with the same number of physical sensors, the MRA performs the best. However, the MRA fails to provide closed-form expressions. In general, the two-level nested array can achieve the most DOFs among the sparse array with closed-form expressions. The expressions of DOFs in the proposed G-FODC with different closed-form sparse arrays are shown as below.

2. Since the parent array and the partner array can be different types of sparse arrays, the proposed G-FODC will have different number of physical sensors. Assuming the parent array and the partner array contain $N_1$ and $N_2$ physical sensors, respectively. Then, the number of the pyhsical sensors in the proposed G-FODC is $N_1 + N_2 - 1$ when the parent array and the partner array are both zero-included, for the parent array is colocated with the partner array at the zero point of array position. While in other cases, the number is $N_1 + N_2$.

3. $\lambda$ is called the scale factor in the proposed structure. In practical applications, $\lambda$ can be chosen as any values in the range of $\lambda \leq 2L_{u1} + 1$. Obviously, a larger scale factor can bring a greater number of uniform one-side DOFs.

4. The proposed G-FODC could have low mutual coupling. For one aspact, one can choose sparse array with low mutual coupling as the parent array and the partner array. For another aspact, the mutual coupling of the partner array can be effectively reduced as the adjacent sensor spacings of partner array are multiplied by an integer $\lambda$.

TABLE II: Example of different structures for comparison

| Array structures | $(N, N_{\text{EXT}})$or $(N_1, N_2, N_{\text{EXT}})$or $(N_1, N_2, N_{\text{EXT1}}, N_{\text{EXT2}})$ | Number of sensors | DOF |
|---|---|---|---|
| FOCA | (2,3,4) | 10 | 241 |
| FONA | (3,4,3) | 10 | 337 |
| G-FODC$_{\text{NA}}$ [1] | (2,3,2,3) | 10 | 341 |
| G-FODC$_{\text{MRA}}$ | (5,6) | 10 | 519 |
| FOCA | (2,3,5) | 11 | 293 |
| G-FODC$_{\text{CA}}$ | (2,3,2,3) | 11 | 297 |

[1] The sub-array is NA in G-FODC. Same with the following representations.

## *B. Comparison with other FODC structures*

When exploited FODC to perform DOA estimation, the traditional NLAs such as CA and NA can not achieve satisfied results. Under this condition, the FOCA and the FONA extending the CA and the NA by adding an extra sub-array respectively are proposed. However, the design of the FOCA and the FONA is lack of universality. The proposed G-FODC provides a generalized method for designing the parent array and the partner array, which can be applied in any NLAs.

In the FOCA, there exists a CA and an extra sub-array. Assume that $N_1$ and $N_2$ represent the number of physical sensors in the first sub-array and the second sub-array of the CA, respectively. The extra sub-array obtains $N_{\text{EXT\_CA}}$ sensors. The DOF of the FOCA is $(2M_{\text{FOCA}} + 1)$ [18], where

$$\begin{aligned} M_{\text{FOCA}} = {} & 4N_1N_2N_{\text{EXT\_CA}} + 3N_1N_2 + 2N_1N_{\text{EXT\_CA}} \\ & - N_2N_{\text{EXT\_CA}} + N_1 - N_2 + N_{\text{EXT\_CA}} - 1. \end{aligned} \tag{14}$$

Similarly, the DOF of the FONA is $(2M_{\text{FONA}} + 1)$ [19], where

$$M_{\text{FONA}} = (3N_{\text{EXT\_NA}} + 2)N_2(N_1 + 1) - 2N_{\text{EXT\_NA}} - 2, \tag{15}$$

where $N_1$, $N_2$ and $N_{\text{EXT\_NA}}$ are the number of physical sensors in the dense sub-array and the sparse sub-array of NA and in the extra sub-array, respectively.

For the proposed G-FODC, many NLAs can be exploited. Here, we take the CA and the NA for example. For convenience, we set the two NLAs $\mathbb{S}_1$ and $\mathbb{S}_2$ as the same type of array. Assume that the parent array $\mathbb{S}_1$ and the partner array $\mathbb{S}_{\text{EXT}}$ are CAs whose number of physical sensors in the first sub-array and the second sub-array of the two CAs are $N_1$, $N_2$ and $N_{\text{EXT1\_CA}}$, $N_{\text{EXT2\_CA}}$, respectively. The DOF of the G-FODC$_{\text{CA}}$ is more than $(2M_{\text{G-FODC}}+1)$, where

$$\begin{aligned}M_{\text{G-FODC}} =& 2N_1N_2-2N_2+1+(4N_1N_2-4N_2+3)\\ & (2N_{\text{EXT1\_CA}}N_{\text{EXT2\_CA}}-2N_{\text{EXT2\_CA}}+1).\end{aligned} \tag{16}$$

Then, consider that the parent array $\mathbb{S}_1$ and the partner array $\mathbb{S}_{\text{EXT}}$ are NAs whose number of physical sensors in the dense sub-array and the sparse sub-array of the two NAs are $N_1$, $N_2$ and $N_{\text{EXT1\_NA}}$, $N_{\text{EXT2\_NA}}$, respectively. The DOF of the G-FODC$_{\text{NA}}$ is more than $(2M_{\text{G-FODC}}+1)$, where

$$\begin{aligned}M_{\text{G-FODC}} =& N_1N_2+N_2-1+(2N_1N_2+2N_2-1)\\ & (N_{\text{EXT1\_NA}}N_{\text{EXT2\_NA}}+N_{\text{EXT2\_NA}}-1).\end{aligned} \tag{17}$$

The comparisons of DOFs are listed in Table. II among the FOCA, the FONA and the proposed G-FODC with NA, MRA and CA as the parent array and the partner array. From the Table II, it is clear that with the same number of physical sensors, the proposed G-FODC could always gain more DOFs compared with other structures. Furthermore, we can also find that using different types of NLAs in the proposed G-FODC will result in distrinct DOFs with the fixed number of physical sensors. Generally, MRA chosen as the parent array and the partner array gains the largest DOFs in the G-FODC, while the co-prime array gains the smallest.

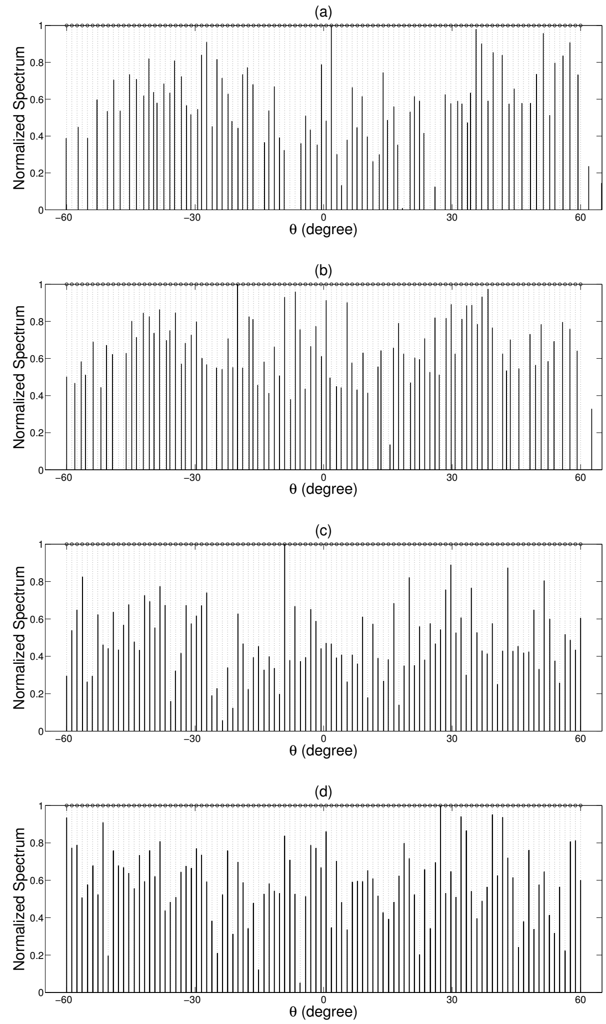


Fig. 2: DOA estimation for different NLA structures. Results for (a)FOCA, (b)FONA, (c)G-FODC$_{\text{NA}}$ and (d)G-FODC$_{\text{MRA}}$.

## IV. Simulation Results

In this section, we present several examples to show the DOA estimation performance of the proposed G-FODC. The DOA estimation of the FOCA and the FONA are shown as constrast. Consider $K$ sources with uniformly distributed directions between $-60^\circ$ and $60^\circ$. The examples have $N$-sensors array with unit spacing half a minimal wavelength.

In the first simulation, a 10-sensor array is considered. The input SNR is 0 dB, the number of signal sources $K$ is 100, and the number of snapshots is 65536. In the 10-sensor array, the parameters are (2,3,4) for the FOCA, (3,4,3) for the FONA, (2,3,2,3) for the G-FODC$_{\text{NA}}$ and (5,6) for the G-FODC$_{\text{MRA}}$.

The results for the four kinds of FODC are shown in Figure 2. The dotted lines represent the actual incident angles of the impinging signals, whereas the solid lines represent the estimation results. Figure 2 has indicated that the G-FODC$_{\text{MRA}}$ and the G-FODC$_{\text{NA}}$ can estimate all the signals accurately, while the FOCA and the FONA fail to implement them all.

In the second simulation, the root mean square error (RMSE) is obtained through 500 Monte Carlo to illustrate the estimation accuracy of the different array structures with 10-sensor array available. First, assume that the number of the sources $K$ is 30. Figure 3(a) gives the results with the SNR varying, where the number of the snapshots is 65536. It is shown that the larger the input SNR is, the higher the estimation accuracy is. With the same input SNR, the proposed G-FODC performs the best among all the structures as expected. Then, we set the input SNR as 0dB and change the number of snapshots. The results are shown in Figure 3(b), where we can see that the estimation accuracy gets better as the number of snapshots increases. Finally, considered that varying signal sources can make the difference illustrated by Figure 3(c). Under the condition of the same SNR and snapshots, the estimation accuracy will decline as the arrays receive more signal sources. These three results of RMSE all indicate that the performance of the proposed G-FODC is the best among all the structures.

## V. Conclusion

This paper presents a generalized approach to design the fourth-order NLA by combining any two second-order NLAs: a parent NLA and a partner NLA. Closed-form expressions for the physical sensor locations and the corresponding virtual

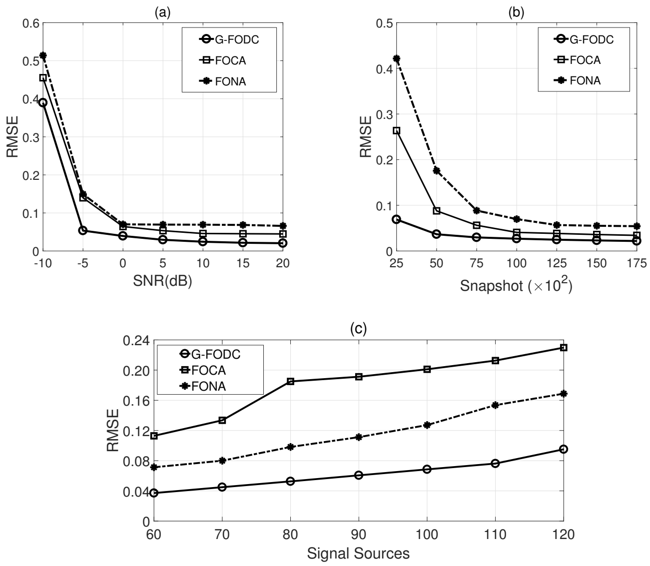


Fig. 3: RMSEs with input SNR, snapshot number and signal source. (a) versus input SNR. (b) versus the number of snapshots. (c) versus signal sources.

sensor configurations are specifically derived. The array design steps are listed one by one. In simulation part, by considering DOA estimation for QS signals, numerical simulation results validate that the proposed geometry could effectively increase the virtual aperture, therefore, having the ability to detect more sources.

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