CRB-based optimal radar placement for target positioning

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***Abstract*—In the radar system, the target estimation performance is affected not only by the measurement error such as the range or angle obtained from the radar but also by the placement of the radar, that is, the geometry. Accordingly, in this paper, we propose a optimal radar placement algorithm that minimizes the Cramer-Rao bound (CRB) by deriving the CRB for the target position in multiple monostatic radar systems with range and angle measurements. In addition, an averaged CRB concept was introduced to improve the estimation performance for successive trajectories. Simulation was performed and favorable results were obtained.**

I. INTRODUCTION

In the radar system, high accuracy range and angle measurements are essential to estimate the position of the target with high accuracy. There have been many attempts to improve the hardware performance of radar and many studies in the field of signal processing to obtain higher position estimation performance even under the same hardware conditions through advanced signal processing [1], [2], [3]. However, the target position estimation performance in radar system has a great influence not only on hardware performance and signal processing ability, but also on relative geometry of sensor and target [4]. This sensor placement research has been studied not only in radar systems [5], [6] but also in fields such as mobile sensor network, etc [7], [8], [9].[6] is the optimal sensor placement using CRB when the time of arrival (ToA) is used as measurement and [9] uses received signal strength (RSS) as measurement. In this study, CRB is derived for target position when range and angle are used as measurements in multiple monostatic radar system. Using the fact that the CRB is related to the placement of the radar, we calculate the optimal placement that minimizes the CRB. Also, the averaged CRB concept is introduced to estimate the radar deployment where the average value of the CRB of the target moving with specific dynamics is minimized.

The contents of this paper are organized as follows. In section 2, there is explanation on target dynamics and the proposed algorithm is described in section 3. In section 4, we conduct the simulation to verify the performance of the proposed algorithm and conclude the paper in section 5.

II. TARGET DYNAMICS

In this study, the target to be estimated by radar system is considered as a missile type that is launched from the launching base as shown in the Fig.1, and is flying with thrust during a certain time according to the shooting range of the target and flying through a free fall in a section without thrust.

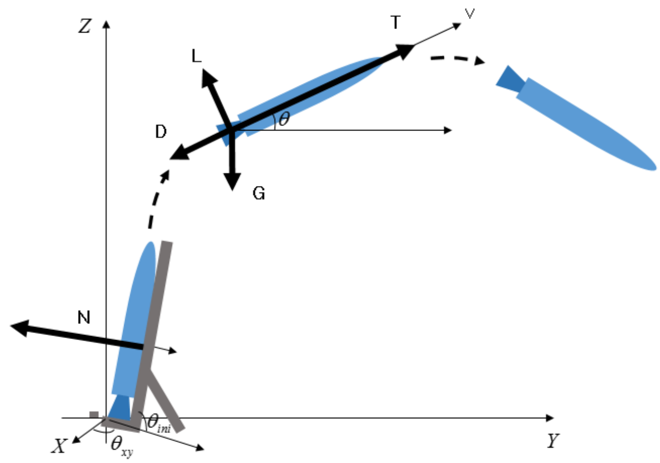

Fig. 1. Target dynamics

In the Fig.1, T, D, L, G and N are thrust, drag, lift, gravity and normal force, respectively. The arrows indicate the direction of the force. Each force can be expressed as follows.

$$N = mg\cos\theta_{ini} \tag{1}$$

$$T = I_{sp}\frac{W_p}{t_b} \tag{2}$$

$$D = \frac{1}{2}\rho v^2 c_D s \tag{3}$$

$$L = \frac{1}{2}\rho v^2 c_L s \tag{4}$$

$$G = mg \tag{5}$$

The normal force is inversely proportional to the angle of the launch pad (θ_{ini}) and occurs only until the target is in contact with the launch pad. m is weight of target and $g = 9.8m/s^2$. The frictional force is ignored. The thrust is proportional to the amount of propellant (W_p) and the specific thrust (I_{sp}) and occurs until the propellant is exhausted. As the amount of fuel decreases over time, the weight of the missile also decreases. t_b is burning time. Drag and lift are proportional to the square of the target velocity (v), the surface area of the target (s), the atmospheric density (ρ) and the lift coefficient (c_L) and the drag coefficient (c_D), respectively. θ_{xy} is the angle in xy-axis. In this study, velocity (v_x, v_y, v_z), position (x, y, z), angle of flight (θ) and target weight (m) are defined as state variables of dynamic equations shown below [10].

$$\begin{bmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \\ \dot{v}_z(t) \end{bmatrix} = \frac{1}{m(t)} \begin{bmatrix} F_x(t) \\ F_y(t) \\ F_z(t) \end{bmatrix} \tag{6a}$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix} \tag{6b}$$

$$\theta(t) = \tan^{-1}\left(v_z(t) / \sqrt{v_x^2(t) + v_y^2(t)} \right) \tag{6c}$$

$$\dot{m}(t) = -\frac{W_p}{t_b} \tag{6d}$$

where $\begin{bmatrix} F_x(t) & F_y(t) & F_z(t) \end{bmatrix}^T$ is

$$\begin{aligned} &\begin{bmatrix} F_x(t) & F_y(t) & F_z(t) \end{bmatrix}^T \\ &= \begin{bmatrix} ((T-D)\cos(\theta(t)) - (N+L)\sin(\theta(t)))\cos\theta_{xy} \\ ((T-D)\cos(\theta(t)) - (N+L)\sin(\theta(t)))\sin\theta_{xy} \\ (T-D)\sin(\theta(t)) + (N+L)\cos(\theta(t)) - mg \end{bmatrix} \end{aligned} \tag{7}$$

In this paper, time update for generating target trajectory is performed by discretizing the above equation. The discretized form of above is omitted.

III. CRB-BASED OPTIMAL RADAR PLACEMENT ALGORITHM

In this study, we find the optimal radar placement that can minimize the CRB of the target position error variance in the environment where the position of the target is estimated from the range and angle measurements obtained from multiple monostatic radars. CRB means the theoretical minimum estimation error variance bound that the estimator can have. For any estimator, CRB is defined as follows [11].

$$\mathrm{var}(\hat{\mathbf{x}}|\,\mathbf{x}) \geq CRB(\mathbf{x}) = FIM^{-1}(\mathbf{x}) \tag{8}$$

$$FIM(\mathbf{x}) = -E\left[\frac{\partial}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{x}^T} \ln p(\mathbf{z}|\,\mathbf{x}) \right] \tag{9}$$

In the above equation, FIM is the Fisher information matrix (FIM) and $\mathbf{z}$, $\mathbf{x}$ and $\hat{\mathbf{x}}$ represent the measurements, variables and estimated variables respectively. The relationship between the measurements of range and angle obtained in multiple monostatic radar system and the position of the target and the radars can be expressed by the following equations.

$$\begin{aligned} r_i &= \sqrt{(x - x_{r,i})^2 + (y - y_{r,i})^2 + (z - z_{r,i})^2} + n_{r,i} \\ &= R_i + n_{r,i} \end{aligned} \tag{10a}$$

$$\begin{aligned} \theta_i &= \tan^{-1}\left(\frac{z - z_{r,i}}{\sqrt{(x - x_{r,i})^2 + (y - y_{r,i})^2}} \right) + n_{\theta,i} \\ &= \vartheta_i + n_{\theta,i} \end{aligned} \tag{10b}$$

In the above equation, $\mathbf{x}_t = (x, y, z)^T$ is the target position and $\mathbf{x}_{r,i} = (x_{r,i}, y_{r,i}, z_{r,i})^T$ is i-th radar position ($i = 1, ..., N_r$). In this paper, we assume $n_{r,i} \sim N(0, \sigma_r^2)$, $n_{\theta,i} \sim N(0, \sigma_\theta^2)$ and measurement noises of different radars are independent. We consider elevation angle measurements only. The $p(r_i|\,\mathbf{x})$ for range obtained from i-th radar becomes

$$p(r_i|\,\mathbf{x}) = \frac{1}{\sqrt{2\pi}\sigma_r} \exp\left(-(r_i - R_i)^2 / (2\sigma_r^2) \right) \tag{11}$$

where $\mathbf{x} = (\mathbf{x}_t, \mathbf{x}_{r,1}, ..., \mathbf{x}_{r.N_r})$. Since we can assume that the range measurements obtained from the total N_r radars are independent, $p(\mathbf{r}|\,\mathbf{x})$ can be expressed as following equation.

$$p(\mathbf{r}|\,\mathbf{x}) = \frac{1}{(2\pi\sigma_r^2)^{\frac{N_r}{2}}} \exp\left(-\sum_{i=1}^{N_r} (r_i - R_i)^2 / (2\sigma_r^2) \right) \tag{12}$$

Where $\mathbf{r}$ is $\mathbf{r} = \begin{bmatrix} r_1 & r_2 & \cdots & r_{N_r} \end{bmatrix}^T$. As above, the probability $p(\boldsymbol{\theta}|\,\mathbf{x})$ for angle can be expressed as

$$p(\boldsymbol{\theta}|\,\mathbf{x}) = \frac{1}{(2\pi\sigma_\theta^2)^{\frac{N_r}{2}}} \exp\left(-\sum_{i=1}^{N_r} (\theta_i - \vartheta_i)^2 / (2\sigma_\theta^2) \right). \tag{13}$$

In the above equation, $\boldsymbol{\theta}$ is $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_{N_r} \end{bmatrix}^T$. Since $\mathbf{r}$ and $\boldsymbol{\theta}$ can be assumed to be independent, $p(\mathbf{z}|\,\mathbf{x}) = p(\mathbf{r}, \boldsymbol{\theta}|\,\mathbf{x})$ becomes $p(\mathbf{z}|\,\mathbf{x}) = p(\mathbf{r}|\,\mathbf{x})\, p(\boldsymbol{\theta}|\,\mathbf{x})$. Therefore, $p(\mathbf{z}|\,\mathbf{x})$ is as follows.

$$\begin{aligned} &p(\mathbf{z}|\,\mathbf{x}) = p(\mathbf{r}|\,\mathbf{x})\, p(\boldsymbol{\theta}|\,\mathbf{x}) \\ &= \frac{1}{(2\pi\sigma_r^2)^{\frac{N_r}{2}} (2\pi\sigma_\theta^2)^{\frac{N_r}{2}}} \exp\left(-\sum_{i=1}^{N_r} \left(\begin{array}{c} (r_i - R_i)^2 / (2\sigma_r^2) \\ +(\theta_i - \vartheta_i)^2 / (2\sigma_\theta^2) \end{array} \right) \right) \end{aligned} \tag{14}$$

To obtain the CRB, we first calculate the FIM. In this problem, since we want to know the CRB with respect to target position, FIM can be expressed as equation below.

$$\begin{aligned} &FIM(\mathbf{x}) = -E\left[\tfrac{\partial}{\partial \mathbf{x}^T} \tfrac{\partial}{\partial \mathbf{x}} \ln p(\mathbf{z}|\,\mathbf{x}) \right] \\ &= FIM_{\mathbf{r}}(\mathbf{x}) + FIM_{\boldsymbol{\theta}}(\mathbf{x}) \\ &= \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} + \begin{bmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{bmatrix} \end{aligned} \tag{15}$$

Since $a_{xy} = a_{yx}, a_{xz} = a_{zx}, a_{yz} = a_{zy}$ and $b_{xy} = b_{yx}, b_{xz} = b_{zx}, b_{yz} = b_{zy}$, we can obtain $FIM(\mathbf{x})$ if we know 6 elements of $FIM_{\mathbf{r}}(\mathbf{x})$ and $FIM_{\boldsymbol{\theta}}(\mathbf{x})$. By partial differentiation, each element of $FIM_{\mathbf{r}}(\mathbf{x})$ and $FIM_{\boldsymbol{\theta}}(\mathbf{x})$ becomes

$$a_{xx} = \frac{1}{\sigma_r^2}\left(\sum_{i=1}^{N_r}\frac{(x-x_i)^2}{R_i^2}\right) \tag{16a}$$

$$a_{yy} = \frac{1}{\sigma_r^2}\left(\sum_{i=1}^{N_r}\frac{(y-y_i)^2}{R_i^2}\right) \tag{16b}$$

$$a_{zz} = \frac{1}{\sigma_r^2}\left(\sum_{i=1}^{N_r}\frac{(z-z_i)^2}{R_i^2}\right) \tag{16c}$$

$$a_{xy} = a_{yx} = \frac{1}{\sigma_r^2}\left(\sum_{i=1}^{N_r}\frac{(x-x_i)(y-y_i)}{R_i^2}\right) \tag{16d}$$

$$a_{xz} = a_{zx} = \frac{1}{\sigma_r^2}\left(\sum_{i=1}^{N_r}\frac{(x-x_i)(z-z_i)}{R_i^2}\right) \tag{16e}$$

$$a_{yz} = a_{zy} = -\frac{1}{\sigma_r^2}\left(\sum_{i=1}^{N_r}\frac{(y-y_i)(z-z_i)}{R_i^2}\right) \tag{16f}$$

$$b_{xx} = \frac{1}{\sigma_\theta^2}\sum_{i=1}^{N_r}\left(\begin{array}{l}\left(\frac{(x-x_i)^2+(y-y_i)^2}{R_i}\right)^2\\ \times\left(\frac{\partial}{\partial x}(\tan\vartheta_i)\right)^2\end{array}\right) \tag{17a}$$

$$b_{yy} = \frac{1}{\sigma_\theta^2}\sum_{i=1}^{N_r}\left(\begin{array}{l}\left(\frac{(x-x_i)^2+(y-y_i)^2}{R_i}\right)^2\\ \times\left(\frac{\partial}{\partial y}(\tan\vartheta_i)\right)^2\end{array}\right) \tag{17b}$$

$$b_{zz} = \frac{1}{\sigma_\theta^2}\sum_{i=1}^{N_r}\left(\begin{array}{l}\left(\frac{(x-x_i)^2+(y-y_i)^2}{R_i}\right)^2\\ \times\left(\frac{\partial}{\partial z}(\tan\vartheta_i)\right)^2\end{array}\right) \tag{17c}$$

$$b_{xy} = \frac{1}{\sigma_\theta^2}\sum_{i=1}^{N_r}\left(\begin{array}{l}\left(\frac{(x-x_i)^2+(y-y_i)^2}{R_i}\right)^2\\ \times\left(\frac{\partial}{\partial x}(\tan\vartheta_i)\right)\left(\frac{\partial}{\partial y}(\tan\vartheta_i)\right)\end{array}\right) \tag{17d}$$

$$b_{xz} = \frac{1}{\sigma_\theta^2}\sum_{i=1}^{N_r}\left(\begin{array}{l}\left(\frac{(x-x_i)^2+(y-y_i)^2}{R_i}\right)^2\\ \times\left(\frac{\partial}{\partial x}(\tan\vartheta_i)\right)\left(\frac{\partial}{\partial z}(\tan\vartheta_i)\right)\end{array}\right) \tag{17e}$$

$$b_{yz} = \frac{1}{\sigma_\theta^2}\sum_{i=1}^{N_r}\left(\begin{array}{l}\left(\frac{(x-x_i)^2+(y-y_i)^2}{R_i}\right)^2\\ \times\left(\frac{\partial}{\partial y}(\tan\vartheta_i)\right)\left(\frac{\partial}{\partial z}(\tan\vartheta_i)\right)\end{array}\right) \tag{17f}$$

where $\frac{\partial}{\partial x}(\tan\vartheta_i)$, $\frac{\partial}{\partial y}(\tan\vartheta_i)$ and $\frac{\partial}{\partial z}(\tan\vartheta_i)$ are calculated as follows.

$$\frac{\partial}{\partial x}(\tan\vartheta_i) = -(x-x_i)(z-z_i)\left((x-x_i)^2+(y-y_i)^2\right)^{-\frac{3}{2}} \tag{18a}$$

$$\frac{\partial}{\partial y}(\tan\vartheta_i) = -(y-y_i)(z-z_i)\left((x-x_i)^2+(y-y_i)^2\right)^{-\frac{3}{2}} \tag{18b}$$

$$\frac{\partial}{\partial z}(\tan\vartheta_i) = \left((x-x_i)^2+(y-y_i)^2\right)^{-\frac{1}{2}} \tag{18c}$$

Resultingly, the CRB with respect to error variance of target position can be obtained by obtaining the inverse matrix of the $FIM(\mathbf{x})$.

$$CRB(\mathbf{x}) = FIM^{-1}(\mathbf{x}) \tag{19}$$

As shown in the $FIM(\mathbf{x})$ derivation, the $CRB(\mathbf{x})$ depends on the placement of the radars. That is, it can be said that the placement of the radar that minimizes the trace of $CRB(\mathbf{x})$ with respect to the target position is theoretically the most optimal arrangement. In this study, we consider moving object, so $CRB(\mathbf{x})$ should be calculated according to all positions of the target. Accordingly, the averaged $CRB(\mathbf{x})$ is defined as follows [11].

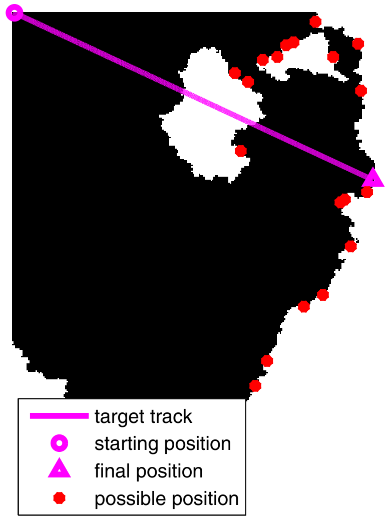

Fig. 2. Simulation region

$$\begin{aligned} & CRB_{avrg}(\mathbf{x}_{r,1},\mathbf{x}_{r,2},\ldots,\mathbf{x}_{r,N_r}) \\ &= \tfrac{1}{T_f-T_s}\int_{T_s}^{T_f} CRB(\mathbf{x}_t(t),\mathbf{x}_{r,1},\mathbf{x}_{r,2},\ldots,\mathbf{x}_{r,N_r})\,dt \\ &\simeq \tfrac{1}{N}\sum_{n=1}^{N} CRB(\mathbf{x}_t(n\Delta t),\mathbf{x}_{r,1},\mathbf{x}_{r,2},\ldots,\mathbf{x}_{r,N_r})\,\Delta t \end{aligned} \tag{20}$$

where T_s, T_f, Δt and N are start time, end time, sampling time and number of samples. $\mathbf{x}_t(t)$ and $\mathbf{x}_t(n\Delta t)$ are target position at time t and $n\Delta t$. Then if we solve equation below, we can obtain optimal radar placement.

$$\hat{\mathbf{x}}_r = \underset{\mathbf{x}_r}{\arg min}\left(tr\left(CRB_{avrg}(\mathbf{x}_r)\right)\right) \tag{21}$$

Where $\mathbf{x}_r$ is $\mathbf{x}_r = (\mathbf{x}_{r,1},\mathbf{x}_{r,2},\ldots,\mathbf{x}_{r,N_r})$.

IV. Simulation Results

To evaluate the performance of the algorithm, we consider artificially created regions (350km ×300km) as shown in Fig.2. In Fig.2, black and white are land and sea, respectively. We assume there are nineteen points where the radar can be installed along the coastline. Radar position candidates are indicated by red dots. The trajectory of the target is marked in magenta, starting at the point marked circle and ending at the point marked triangle. The trajectory parameter and initial state variables of the target are shown in the table I.

Range and elevation angle measurements are used and each measurement error is set to 5m and 1°, respectively.

TABLE I. SIMULATION PARAMETERS AND INITIAL STATES

Parameter	Value	initial state	Value
I_{sp}	2000	x	0m
t_b	30sec	y	0m
c_D	0.4	z	0.1m
c_L	0.4	v_x	0m/s
W_p	100kg	v_y	0m/s
Δt	0.1sec	v_z	0m/s
θ_{xy}	65°	m	200kg
		θ	60°

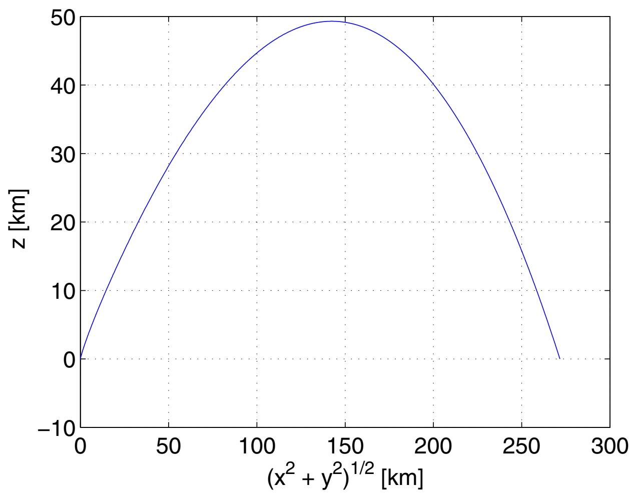

Fig. 3. Target trajectory

In order to analyze the performance of the proposed method according to the number of radar, simulation was performed in the environments where the number of radar is 3, 4, and 5. The Fig.3 shows the trajectory of the target obtained as a result of table 1 in 2D. The trajectory was terminated when the altitude was less than zero, and the flight took about 215 seconds. The results of the proposed algorithm are shown in the case where the number of radar is 3, 4 and 5, respectively. In Fig.4, 5 and 6, the blue squares and green circles indicate the radar position with the lowest and largest trace of averaged CRB, respectively. As can be seen with the naked eye, the radar placement marked blue square has a broad distribution, that is, a low position dilution of precision (PDOP), as compared with green circles. The table II shows the minimum averaged CRB for each axis according to the number of radar.

V. CONCLUSION

In this study, we derived the CRB when estimating the target with the range and angle measurements and found the placement of the radar with the lowest trace of averaged CRB in moving target estimation. The CRB is lowered as the number of radars increases, and the difference in PDOP between the radar placement minimizing the CRB and maximizing the CRB is also visually observed. In this paper, we apply the averaged CRB which averages the CRB for all the positions of the target, but we will consider the idea of weighting the CRB for the section where the target tracking is relatively important in the trajectory for further work.

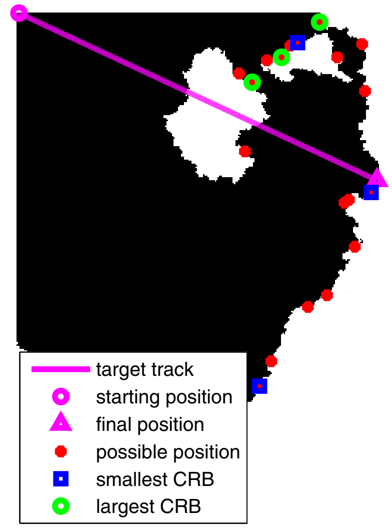

Fig. 4. Results with 3 radars

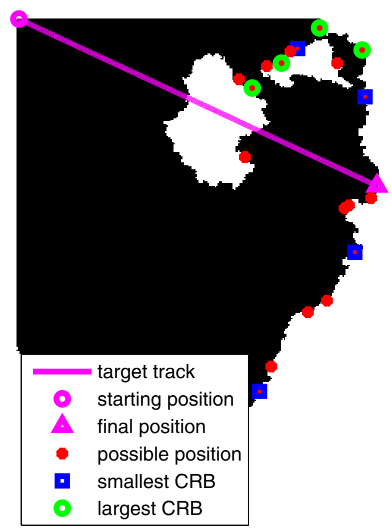

Fig. 5. Results with 4 radars

TABLE II. RESULTS OF MINIMUM AVERAGED CRB

Number of radar	$(CRB_{avrg,x}, CRB_{avrg,y}, CRB_{avrg,z},)$
3	$(35.98, 21.00, 74.60)\ m^2$
4	$(21.15, 15.28, 42.85)\ m^2$
5	$(18.27, 12.84, 36.43)\ m^2$

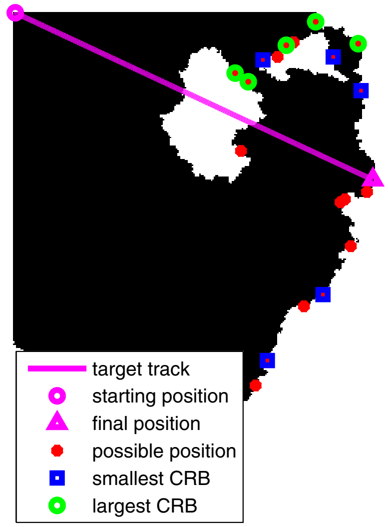

Fig. 6. Results with 5 radars

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