Hybrid Method of DOA Estimation Using Nested Array for Unequal Power Sources

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***Abstract*—In the case of unequal power sources, due to the existence of strong sources, the performance of the subspace or compressive sensing (CS) based algorithms using a nested array suffers serious degradation for directions of arrival (DOAs) estimation of weak sources. In this paper, a hybrid estimation method is proposed to overcome this problem, and meanwhile is used to deal with underdetermined DOA estimation cases which are achieved by the nested array. In this hybrid method, CS-based algorithm is introduced first to estimate the DOAs of strong sources, which depends on the virtual uniform linear array (ULA) obtained from a nested array. Second, a covariance matrix with increased degrees of freedom, which is restored from the virtual ULA by Toeplitz matrix, is used to remove the effect of strong sources by orthogonal complement. Finally, the covariance matrix without strong sources can be used for estimating the DOAs of weak sources by MUSIC algorithm. Simulation results demonstrate the superior performance of the proposed method in terms of DOA estimation for unequal power sources.**

Keywords*—Unequal power sources, nested array, DOA estimation, compressive sensing, orthogonal complement.

I. Introduction

In the last few years, direction of arrival (DOA) estimation using a uniform linear array (ULA) is a hot issue in the area of signal processing [1]. It is well known that the classic subspace-based methods for DOA estimation, such as MUSIC [2] and ESPRIT [3], can identify up to $N-1$ sources with a conventional N-antenna ULA. To increase the number of detected sources, a nested array [4] has been proposed to increase the degrees of freedoms (DOFs), which is provided by a virtual long ULA in the difference co-array from vectoring the covariance matrix. Particularly, it can obtain $O\left(N^2\right)$ DOFs with only $O\left(N\right)$ elements and then can resolve dramatically more uncorrelated sources than the actual number of physical elements.

Accordingly, based on such sparse array configuration, the subspace-based algorithms have been first implemented for underdetermined DOA estimation following spatial smoothing [4] or Toeplitz matrix [5]. Note that, because of vectoring covariance matrix (as a second-order statistics), the DOA estimation algorithms based on the obtained virtual ULA with $O\left(N^2\right)$ DOFs need to deal with the cases of coherent signals and a single snapshot. The traditional subspace-based algorithms can not be implemented directly. Thus, the spatial smoothing [4] or Toeplitz matrix [5] has been introduced to handle signal decorrelation and recover the full rank of the resulting data covariance matrix. However, the application of spatial smoothing or Toeplitz matrix leads to a significant DOF reduction of aforementioned virtual ULA, thus the DOA estimation performance is compromised [6].

Recently, compressive sensing (CS) for DOA estimation, which explores the sparsity of the original signal, has been an attractive research field [7]. This is because that CS-based algorithm not only provides super resolution for sparse signals, but also can deal with the case of single snapshot or coherent signals, which is in coincide with the condition of the obtained virtual ULA. In [8], the CS-based methods (e.g., $l1$-norm regularized least squares) have been used to estimate DOA for a nested array and have better performance than subspace-based methods.

In practice, however, both the electromagnetic wave powers of different sources, and the distances between the passive radar and these sources are different, which lead to the fact that the sources received by the passive radar are unequal-powered. In this case, the performance of subspace and CS based methods may suffer serious degradation [9]. This problem is not the first to be considered. Generally, there are two major methods for solving the problem: one is to resolve the weak sources after mitigating strong sources in spatial domain [10-11]; the other is to simultaneously resolve weak and strong sources with certain modification such as iteration [12-13]. However, the former needs to know the DOAs of strong sources, which is difficult to be known apriori in reality. The latter has increased error in eigenvectors of weak sources due to existence of strong sources.

In this paper, based on the nested array, we will propose a hybrid method to enhance the DOA estimation performance in the case of unequal power sources, while still preserving all advantages of the nested array. As already mentioned, the virtual ULA obtained after vectoring the covariance matrix of the receive data can not only be used for CS-based algorithms, but also can be regarded as a middle step for restoring a full rank covariance matrix in order to apply subspace-based algorithms. Thus we can first estimate DOAs of strong sources by a CS-based algorithm. Then the full rank covariance matrix is created, and orthogonal complement is introduced to suppress strong sources and maintain weak sources. Finally, the full rank covariance matrix without strong sources is used to estimate the DOAs of weak sources by a subspace-based algorithm.

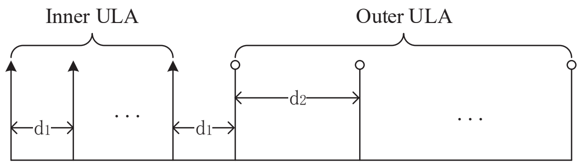

Fig. 1: A two-level nested scale-sensor array composed of inner ULA with N_1 antennas and outer ULA with N_2 antennas.

The rest of this paper is organised as follows. Section II reviews signal model. In Section III, the hybrid method is proposed in details. Then, numerical examples are given to show the effectiveness of the proposed method in Section IV. Finally, Section V concludes this paper.

II. Signal model

A two-level nested array with N-antennas [4] consists of two concatenated ULAs, as shown in Fig. 1. Specifically, the inner ULA has N_1 antennas with antenna distance d_1, and the outer ULA has N_2 antennas with antenna distance $d_2 = (N_1+1)\,d_1$. So $N = N_1 + N_2$. We consider that K far-field narrowband signals are impinging on this array from directions $\{\phi_k, k = 1, 2, \ldots, K\}$. The data model $\boldsymbol{y}_{na}$ of a nested array at time t is give as

$$\boldsymbol{y}_{na}(t) = \boldsymbol{A}_{na}\boldsymbol{x}(t) + \boldsymbol{n}(t), \tag{1}$$

where the nested array manifold matrix $\boldsymbol{A}_{na} = [\boldsymbol{a}(\phi_1)\ \boldsymbol{a}(\phi_2)\ \ldots\ \boldsymbol{a}(\phi_K)]$ with

$$\boldsymbol{a}(\phi_i) = \big[e^{-j2\pi\sin\phi_i d_1/\lambda} \quad \ldots \quad e^{-j2\pi\sin\phi_i d_1 N_1/\lambda} \quad e^{-j2\pi\sin\phi_i d_1(N_1+1)/\lambda} \quad \ldots \quad e^{-j2\pi\sin\phi_i d_1(N_1+1)N_2/\lambda}\big]^{\mathrm{T}}, \tag{2}$$

where λ is wave length, and $\{\cdot\}^{\mathrm{T}}$ denotes transpose operator. Furthermore, the white Gaussian noise $\boldsymbol{n}(t)$ is assumed to be uncorrelated from the signals. Also, the signals are assumed to be temporally white and uncorrelated from each other. Following these assumptions, the autocorrelation matrix of $\boldsymbol{y}_{na}$ can be given as

$$\boldsymbol{R_y} = \mathrm{E}\{\boldsymbol{y}_{na}\boldsymbol{y}_{na}^{\mathrm{H}}\} = \boldsymbol{A}_{na}\boldsymbol{R_x}\boldsymbol{A}_{na}^{\mathrm{H}} + \sigma_n^2\boldsymbol{E}_{N\times N}, \tag{3}$$

where $\boldsymbol{R_x} = \mathrm{diag}\left[\sigma_1^2\ \sigma_2^2\ \ldots\ \sigma_K^2\right]$ is the signal covariance matrix with σ_i^2 being the ith signal's power, σ_n^2 is the noise power and $\boldsymbol{E}_{N\times N}$ is an $N\times N$ identity matrix. $\mathrm{E}\{\cdot\}$ and $(\cdot)^{\mathrm{H}}$ denote mathematical expectation and conjugate transpose, respectively.

Vectorizing $\boldsymbol{R_y}$, we have

$$\boldsymbol{Z} = (\boldsymbol{A}_{na}^{*} \odot \boldsymbol{A}_{na})\,\boldsymbol{s} + \sigma_n^2\boldsymbol{E}_n, \tag{4}$$

where $\odot$ denotes the Khatri-Rao (KR) product, $(\cdot)^{*}$ denotes conjugation, $\boldsymbol{s} = \left[\sigma_1^2\ \sigma_2^2\ \ldots\ \sigma_K^2\right]^{\mathrm{T}}$, and $\boldsymbol{E}_n = [\boldsymbol{e}_1\ \boldsymbol{e}_2\ \ldots\ \boldsymbol{e}_K]^{\mathrm{T}}$, with $\boldsymbol{e}_i$ being a row vector of all zeros except a "1" at the ith position. Here, for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same number of columns, the definition of KR product is provided as

$$\boldsymbol{A}\odot\boldsymbol{B} = [\boldsymbol{a}_1\otimes\boldsymbol{b}_1, \boldsymbol{a}_2\otimes\boldsymbol{b}_2,\ \cdots\ , \boldsymbol{a}_k\otimes\boldsymbol{b}_k], \tag{5}$$

where $\otimes$ denotes the Kronecker product.

It can be seen that $\boldsymbol{Z}$ in (4) can be regarded as a new longer received data with a new array manifold matrix $\boldsymbol{A}_{na}^{*}\odot\boldsymbol{A}_{na}$ and the new signal vector $\boldsymbol{s}$ composed of signal powers instead of signal amplitudes.

III. Hybrid method of DOA estimation for unequal power sources

In order to perform subspace-based DOA estimation algorithms, we assume that the number of sources is known apriori or can be estimated (e.g., by identifying the rank of source subspace [14]).

First, we consider solving (4) in terms of the sparse signal recovery through CS. Here, the performance of CS-based algorithm will be improved due to the fact that the aperture of virtual ULA (e.g., $(N^2-1)/2+N$ for $N_1 = N_2$) is larger than that of ULA with N antennas. The desired result of $\boldsymbol{s}$ represents the solution to the following constrained minimization problem:

$$\begin{aligned}&\tilde{\boldsymbol{s}} = \arg\min_{\boldsymbol{s}} \|\boldsymbol{s}\|_0\\ &\text{s.t. } \left\|\boldsymbol{Z} - (\boldsymbol{A}_s^{*}\odot\boldsymbol{A}_s)\,\boldsymbol{s} - \sigma_n^2\boldsymbol{E}_n\right\|_2 < \delta.\end{aligned} \tag{6}$$

Here, $\|\cdot\|_0$ and $\|\cdot\|_2$ denote the l_0 norm and Euclidean (l_2) norm of a vector, respectively, and δ is a user-specific bound. This type of problems has been the main objective of studies in the field of CS. Several numerical computation methods have been investigated. In this step, we consider the Lasso-based method for solving the problem, which is similar to the Lasso-based algorithm used for DOA estimation based on a co-prime array in [6]. Note that other methods may also be used instead of the Lasso-based algorithm. According to the Lasso-based technique for DOA estimation in [6,15], the strong sources are estimated. It should be noted that the Lasso-based spectrum estimation technique may induce spurious peaks especially when signal-to-noise ratio (SNR) is low or the number of snapshots is small. Since there is no necessary to know the number of strong sources in our method, the DOA estimation can be empirically obtained by selecting significantly dominant peaks of the spatial spectrum to avoid the effect of spurious peaks. The DOAs of sources, which are not determined in this step, can be estimated in the following processing.

Second, since Toeplitz matrix has less computational burden compared with spatial smoothing technique, it will be used to restore full rank covariance matrix $\boldsymbol{R}_z$ from (4) after removing repeated items (after their first appearance) [4]. In doing so, the rank of $\boldsymbol{R}_z$ is $\left(N^2/4+N/2\right)$. Then, according to the estimated DOAs of strong sources, a block matrix $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} = [\ \tilde{\boldsymbol{a}}(\phi_1) \quad \cdots \quad \tilde{\boldsymbol{a}}(\phi_L)\ ], \tag{7}$$

where L is the number of strong sources which are estimated by the Lasso-based algorithm, and $\tilde{\boldsymbol{a}}(\phi_i) = \left[\ e^{-j2\pi\sin\phi_i d_1/\lambda} \quad \ldots \quad e^{-j2\pi\sin\phi_i d_1\left(N^2/4+N/2\right)/\lambda}\ \right]^{\mathrm{T}}$ is the steering vector of the ith strong source. Note that the

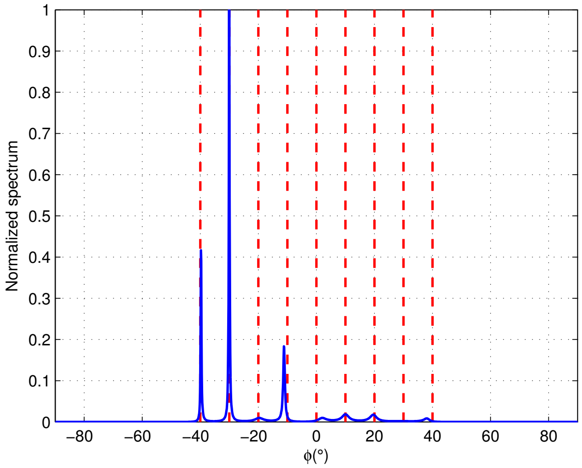

Fig. 2: Spectrum of MUSIC algorithm.

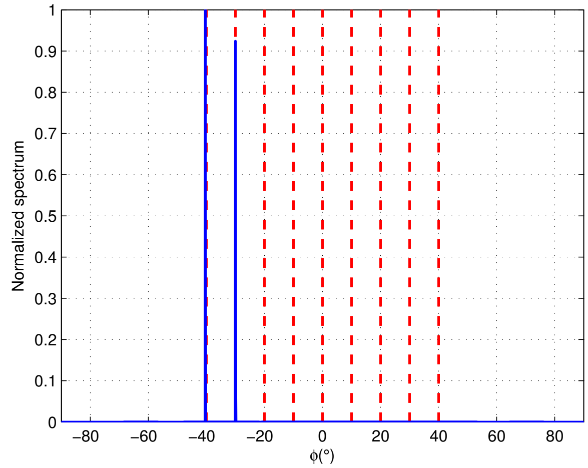

Fig. 3: Spectrum of Lasso-based algorithm.

length of $\tilde{\boldsymbol{a}}\left(\phi_i\right)$ equals to the rank of $\boldsymbol{R}_z$. Therefore, the orthogonal complement of $\boldsymbol{B}$ is given as

$$\boldsymbol{P}_{\boldsymbol{B}}^{\perp} = \boldsymbol{E}_{\left(N^2/4+N/2\right)\times\left(N^2/4+N/2\right)} - \boldsymbol{B}\left(\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{\mathrm{H}}. \quad (8)$$

Considering (8), a covariance matrix $\boldsymbol{R}_{zz}$ after removing the estimated strong sources by $\boldsymbol{P}_{\boldsymbol{B}}^{\perp}$ can be obtained as

$$\boldsymbol{R}_{zz} = \boldsymbol{P}_{\boldsymbol{B}}^{\perp}\boldsymbol{R}_z\left(\boldsymbol{P}_{\boldsymbol{B}}^{\perp}\right)^{\mathrm{H}}. \quad (9)$$

Here, $\boldsymbol{R}_{zz}$ can be considered as an approximated matrix for a output $\boldsymbol{R}_{out}$ of spatial filter by $\boldsymbol{P}_{\boldsymbol{B}}^{\perp}$ due to finite sample size. $\boldsymbol{R}_{out}$ can be given as

$$\boldsymbol{R}_{out} = \mathrm{E}\left\{\boldsymbol{P}_{\boldsymbol{B}}^{\perp}y\left(t\right)\left(\boldsymbol{P}_{\boldsymbol{B}}^{\perp}y\left(t\right)\right)^{\mathrm{H}}\right\} = \boldsymbol{P}_{\boldsymbol{B}}^{\perp}\boldsymbol{R}_{yy}\left(\boldsymbol{P}_{\boldsymbol{B}}^{\perp}\right)^{\mathrm{H}}, \quad (10)$$

where $\boldsymbol{R}_{yy} = \mathrm{E}\left\{y\left(t\right)\left(y\left(t\right)\right)^{\mathrm{H}}\right\}$ and

$$\boldsymbol{y}\left(t\right) = \boldsymbol{A}\boldsymbol{x}\left(t\right) + \boldsymbol{n}\left(t\right), \quad (11)$$

with $\boldsymbol{A} = [\tilde{\boldsymbol{a}}(\phi_1)\ \tilde{\boldsymbol{a}}(\phi_2)\ \ldots\ \tilde{\boldsymbol{a}}(\phi_K)]$. In doing so, $\boldsymbol{R}_{zz}$ does not contain the estimated strong sources and maintains the weak sources.

Finally, MUSIC is used to estimate the DOAs of weak sources based on $\boldsymbol{R}_{zz}$ [4]. This processing can not be affected by strong sources, which is a main advantage compared with the mentioned methods that simultaneously resolve weak and strong sources with certain modification [12-13].

Here, we summarize the hybrid method of DOA estimation for unequal power sources. In order to increase DOF, a virtual ULA $\boldsymbol{Z}$ is obtained in the difference co-array constructed from a nested array. In this case, CS-based algorithm (e.g., Lasso-based) as the first step of the proposed method is used for estimating DOAs of strong sources. Subsequently, in the second step of the proposed method, a covariance matrix $\boldsymbol{R}_z$ with increased DOF, which is recovered from the virtual ULA, is used to remove the effect of strong sources by $\boldsymbol{P}_{\boldsymbol{B}}^{\perp}$. Then, in the last step, the covariance matrix without strong sources, i.e., $\boldsymbol{R}_{zz}$, can be used for estimating the DOAs of weak sources by subspace-based algorithm (e.g., MUSIC). It should be noted that due to the similar processing for increasing DOFs in the difference co-array, it is simple to modify our proposed method to deal with underdetermined DOAs estimation for co-prime arrays [16], super nested array [17] and augment nested array [18].

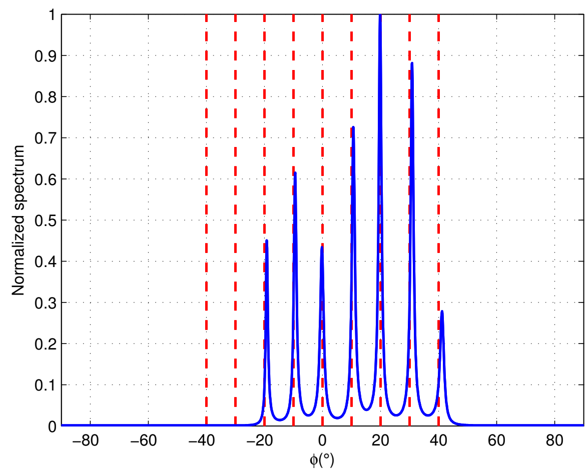

Fig. 4: Spectrum of MUSIC algorithm after removing strong sources.

IV. Simulation Results

In this section, we give numerical examples to show the superior performance of the proposed hybrid method compared with MUSIC [4] and Lasso-based algorithm [6]. In the examples, the nested array contains $N = 6$ antennas with $N_1 = N_2 = 3$. The antenna positions of this nested array are located at $[0\ 1\ 2\ 3\ 7\ 11]\,d$. So the aperture of the obtained virtual ULA and the rank of restored matrix $\boldsymbol{R}_z$ are $\left(N^2-2\right)/2+N = 23$ and $\left(N^2/4+N/2\right) = 12$, respectively. We consider 9 narrowband sources impinging on the nested array from angles uniformly distributed between -40° and

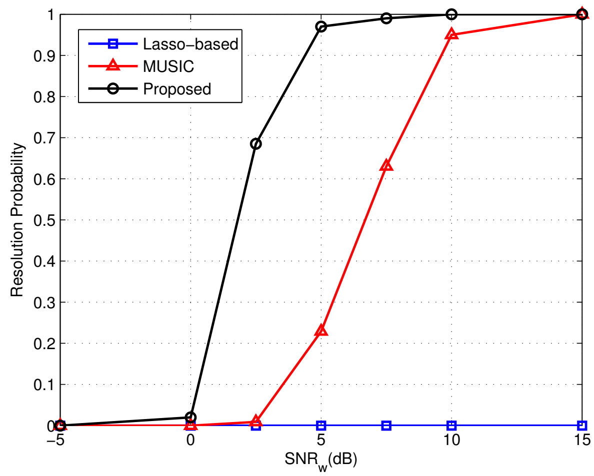

Fig. 5: Resolution probability versus SNR_w.

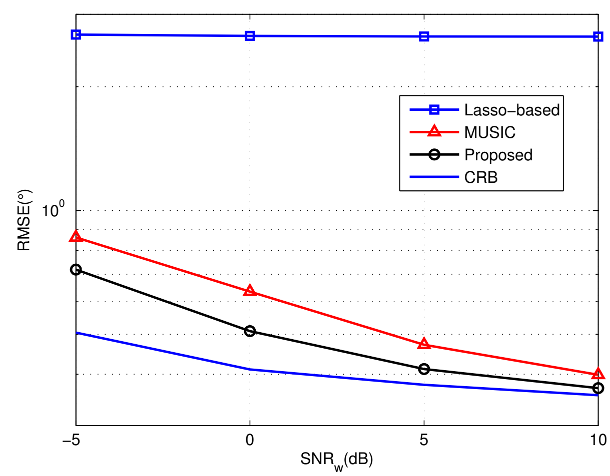

Fig. 6: RMSE versus SNR_w.

$40°$. Without loss of generality, we assume that the signals from two angles on the left (i.e., $-40°$ and $-30°$) are strong sources. The number of snapshots is 800.

A. Spatial Spectrum

Here, the SNR of strong source powers is named as SNR_s and is set as 10 dB, whereas the SNR of weak source powers is named as SNR_w and is set as 0 dB. Figs. 2, 3 and 4 depict normalized spectrums of MUSIC for simultaneously estimating strong and weak sources, Lasso-based algorithm for simultaneously estimating strong and weak sources, and MUSIC for estimating weak sources after removing strong sources, respectively. In Fig. 2, red dotted lines represent true values of DOAs, and so do other figures. The peaks of strong sources in Fig. 2 are much sharper than those of weak sources, and peaks of some weak sources can not be resolved if they exist. This is because in the case of simultaneously resolving strong and weak sources, the weak sources can not be detected easily in the estimation processing due to existence of strong sources. In Fig. 3, the peaks of strong sources are too sharp and can be easily observed in the whole spectrum, whereas the DOAs of weak sources can not be estimated. As far as the proposed hybrid is concerned, Fig. 3 demonstrates the output of the first DOA estimation algorithm (i.e., Lasso algorithm), which has obtained the DOAs of strong sources. And then, Fig. 4 demonstrates seven peaks in the whole spectrum, which indicates that the second DOA estimation algorithm (i.e., MUSIC algorithm) is able to correctly identify the weak sources.

B. Resolution Probability

In order to further compare the performance of DOA estimation algorithms, we provide the comparison of their resolution probabilities for both strong and weak sources versus SNR_w, shown in Fig. 5. Here, the SNR_s is fixed on 20 dB, the number of Monte Carlo simulations is 1000, and the other simulation conditions are the same as those in subsection A. The resolution probability is calculated by the percentage of correct source resolution among Monte Carlo trials. Here, the threshold is set to $3°$. When the difference between estimated angle and true angle is smaller than the threshold, the true angle can be considered to be estimated successfully.

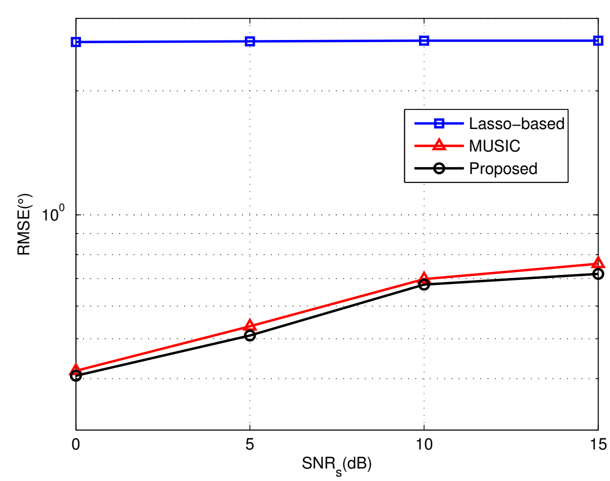

Fig. 7: RMSE versus SNR_s.

In Fig. 5, the blue line with square, which represents the resolution probability of the Lasso algorithm, lies close to zero for all SNR_w, indicating that Lasso-based algorithm can hardly achieve the correct source enumeration. Then, comparing the red strangle line (represents MUSIC) with black circle line (represents proposed hybrid method), it can be seen that the proposed hybrid method achieves a higher enumeration accuracy than MUSIC algorithm especially when SNR_w is larger than 0 dB. These observations illustrate that the proposed hybrid method has superior performance in the case of unequal power sources.

C. *RMSE versus* SNR

For Fig. 6, the SNR_s is fixed on 15 dB, the number of Monte Carlo simulations is 400, and the other simulation conditions are the same as those in the subsection B. The RMESs for both strong and weak sources corresponding to different SNR_w are illustrated in Fig. 6. The corresponding CRB [19,20] is also depicted. It can be seen that the black circle line (represents proposed method) is lower than the blue square line (represents Lasso-based) and the red strangle line (represents MUSIC) for all SNR_w, which indicates that the performance of the proposed method is better than performance of the Lasso-based and MUSIC algorithms. Particularly, the performance of the Lasso-based algorithm is related with the threshold for all SNR_w due to its poor resolution probability. Also, the performance of the proposed method is gradually close to CRB with increased SNR_w. Furthermore, it should be noted that in the case of underdetermined DOA estimation, performance of the proposed method and MUSIC algorithm saturates gradually with increased SNR_w, which is in accordance with what was observed experimentally in [21].

For Fig. 7, the SNR_w is fixed on -5 dB and the other simulation conditions are the same as those in Fig. 6. The RMESs for both strong and weak sources corresponding to different SNR_s are illustrated in Fig. 7. In this case, The strong source powers increase with increased SNR_s, whereas the weak source powers and noise power are fixed. Therefore, with increased SNR_s, the performance of the proposed method and the MUSIC algorithm degrade due to gradually increased difference between strong source power and weak source power. In addition, the performance of the Lasso-based algorithm is related with the threshold for all SNR_s because of its poor resolution probability. Furthermore, one can see that the performance of the proposed method is better than performance of the Lasso-based and MUSIC algorithms in this case.

V. Conclusion

In this paper, we propose a hybrid method to estimate DOAs of unequal power sources for the nested array. We first estimate the DOAs of strong sources by CS-based algorithm, and then identify weak sources after removing the effect of strong sources. Meanwhile, all advantages of the nested array have been preserved. The superior performance of the proposed method has been verified by simulation results. Future research in this direction will be to modify the proposed method used for nested vector-sensor arrays (e.g., nested polarization sensitive arrays).

VI. Acknowledgment

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