A Frequency Estimator for Real Valued Sinusoidal Signals Using Three DFT Samples

Hao Chen, Fangxing Xu and Jiangyuan Li
Key Lab of Aperture Array and Space Application
East China Research Institute of Electronic Engineering
Hefei, China
ch89118@sina.com

***Abstract*—This paper presents a simple frequency estimator for real valued sinusoidal signals using only three DFT samples. Simulations confirm that our proposed method outperforms traditional alternatives based on three DFT samples. Compared with estimators for complex sinusoidal signals, our method increases the estimation accuracy thanks to its consideration of negative spectrum components. Compared with the estimators with windowing technique, our method also has better performance in spite of lower complexity.**

Keywords—DFT; frequency estimation; real valued sinusoidal signals

I. Introduction

Frequency-modulated continuous-wave (FMCW) radars have a large range of applications in the civilian and military sectors, including aircraft navigation (radio altimeters), subsurface examination, weather monitoring, vehicle radars and many more remote sensing applications [1]. In FMCW radars, the frequency difference between the transmitted signal and the received echo signal is proportional to the time of flight (ToF). By mixing the received and transmitted signal, the frequency difference is extracted, from which the target range can be calculated, so estimating the beat frequency is an important procedure in FMCW radar.

The simplest method for frequency estimation is to use N-point discrete Fourier transform (DFT) after sampling a signal. Several frequency estimation methods based on DFT have been proposed. In [2], a two-stage search was implemented to improve the frequency estimate. First of all, a coarse search with an N-point DFT is executed and then a fine search is implemented around the vicinity of the peak determined in the first stage. Furthermore, in order to simplify the second stage fine estimation, several efficient methods were proposed [3]-[7]. In these methods, the DFT bin with maximum magnitude and its immediate left and right neighbors are used to estimate the fine frequency. These methods require very few operations and produce a fine estimate for the frequency.

However, these methods are only effective for complex sinusoidal signals, they are inaccurate for real valued sinusoidal signals because of extra negative frequency components in real valued signals. A straightforward way to minimize the effects of negative spectrum leakage is to multiply the received signal by a window function. In many applications, the DFT calculation is implemented with a properly selected window to suppress the interference caused by undesired spectrum components [8]-[11]. In [8], Belega proposed a method using an analytical expression based on a suitable weighting of the three largest DFT spectrum samples, it is highly effective in rejecting the detrimental effect on the estimation accuracy due to the image component of the signal spectrum. In [11], Borkowski presented a method which can determine the frequency of the fundamental sinusoidal component of a multifrequency signal with only a short measurement. An important feature of this algorithm is the elimination of the impact associated with the negative spectrum component on the estimation's outcome.

In this paper, we have proposed a simple frequency estimation method for real valued sinusoidal signals using three DFT samples, which can suppress the negative frequency interference with no need for windowing techniques. Compared with the traditional frequency estimation methods using window functions, our method has a better performance while with a lower complexity.

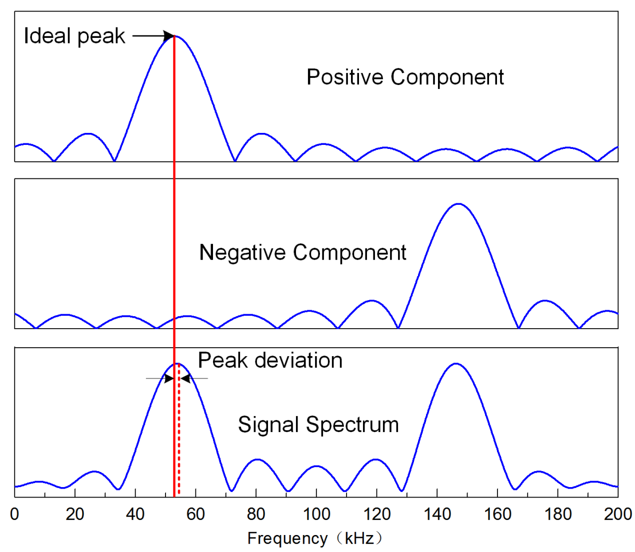

Fig. 1. Spectrum of positive component, negative component and real valued signal.

II. PROBLEM DESCRIPTION

The discrete-time sine-wave observed under white Gaussian noise can be modeled as

$$r[n] = A\cos(2\pi f_0 \frac{n}{f_s} + \varphi_0) + w[n] \quad (1)$$

where f_0 is the signal frequency, f_s is the sampling frequency, A is the amplitude and φ_0 is the initial phase. It is assumed that f_s exceeds $2f_0$ in order to satisfy the Nyquist theorem. The real valued signal in Eq. (1) can be also represented as a superposition of two complex-valued sinusoidal modes as follows:

$$r[n] = \frac{A}{2}e^{j(2\pi f_0 \frac{n}{f_s}+\varphi_0)} + \frac{A}{2}e^{-j(2\pi f_0 \frac{n}{f_s}+\varphi_0)} + w[n] \quad (2)$$

Accordingly, in the discrete frequency domain, the N-point DFT of $r[n]$ is:

$$\begin{aligned} R[k] &= \sum_{n=0}^{N-1}[\frac{A}{2}e^{j(2\pi f_0 \frac{n}{f_s}+\varphi_0)} + \frac{A}{2}e^{-j(2\pi f_0 \frac{n}{f_s}+\varphi_0)}]e^{-j2\pi k\frac{n}{N}} + W[k] \\ &= \frac{A}{2}e^{j[\varphi_0+\pi(\mathrm{N}-1)(\frac{f_0}{f_s}-\frac{k}{N})]}\cdot\frac{\sin[\mathrm{N}\cdot\pi(\frac{f_0}{f_s}-\frac{k}{N})]}{\sin[\pi(\frac{f_0}{f_s}-\frac{k}{N})]} \\ &+ \frac{A}{2}e^{-j[\varphi_0+\pi(\mathrm{N}-1)(\frac{f_0}{f_s}+\frac{k}{N})]}\cdot\frac{\sin[\mathrm{N}\cdot\pi(\frac{f_0}{f_s}+\frac{k}{N})]}{\sin[\pi(\frac{f_0}{f_s}+\frac{k}{N})]} + W[k] \end{aligned} \quad (3)$$

$W[k]$ is Gaussian noise sample in the frequency domain. From Eq. (3), it can be found the spectrum of a real valued sinusoid contains both positive and negative spectrum components. As shown in Fig. 1, the peak of a continuous spectrum for the positive mode is exactly located at the true frequency. However, when added with the spectrum component of the negative mode, due to the spectrum leakage, the peak of the resulting continuous spectrum of the real valued signal $r[n]$ cannot be guaranteed to be at the true frequency any more.

Lots of low complexity methods to estimate the frequency of a complex signal have been proposed, but they are not suitable for the real valued signal. Some literatures make use of a windowing technique to suppress the unwanted spectrum interference. However, such a windowing method increases the computational complexity and reduces the frequency resolution due to main lobe widening. Therefore, a simple method is necessary which can suppress the effect of the negative spectrum leakage without using a window function.

III. PROPOSED ESTIMATOR

We know that the location of a sample with the largest magnitude must be close to the frequency of the signal. Therefore, the signal frequency can be further written as $f_0 = f_s \cdot (k_p + \delta)/\mathrm{N}$, where k_p denotes the index of the DFT sample with the largest magnitude and δ $(-0.5 \le \delta < 0.5)$ represents the fractional frequency part. Our goal in this paper is to estimate δ utilizing three DFT samples around the peak.

According to Eq. (3), we can express the DFT bin where the peak occurs and its immediate left and right neighbors as follows:

$$\begin{aligned} R[k_p] &= \frac{A}{2}e^{j[\varphi_0+\pi(\mathrm{N}-1)\frac{\delta}{N}]}\cdot\frac{\sin(\pi\cdot\delta)}{\sin(\frac{\pi\cdot\delta}{N})} \\ &+ \frac{A}{2}e^{j[\frac{2k_p\pi}{N}-\pi(\mathrm{N}-1)\frac{\delta}{N}-\varphi_0]}\cdot\frac{\sin(\pi\cdot\delta)}{\sin[\frac{\pi\cdot(2k_p+\delta)}{N}]} + W[k_p] \end{aligned} \quad (4)$$

$$\begin{aligned} R[k_p-1] &= \frac{A}{2}e^{j[\varphi_0+\pi(\mathrm{N}-1)\frac{\delta}{N}-\frac{\pi}{N}]}\cdot\frac{\sin(\pi\cdot\delta)}{\sin[\frac{\pi\cdot(\delta+1)}{N}]} \\ &+ \frac{A}{2}e^{j[\frac{2k_p\pi}{N}-\pi(\mathrm{N}-1)\frac{\delta}{N}-\varphi_0-\frac{\pi}{N}]}\cdot\frac{\sin(\pi\cdot\delta)}{\sin[\frac{\pi\cdot(2k_p+\delta-1)}{N}]} + W[k_p-1] \end{aligned} \quad (5)$$

$$\begin{aligned} R[k_p+1] &= \frac{A}{2}e^{j[\varphi_0+\pi(\mathrm{N}-1)\frac{\delta}{N}+\frac{\pi}{N}]}\cdot\frac{\sin(\pi\cdot\delta)}{\sin[\frac{\pi\cdot(\delta-1)}{N}]} \\ &+ \frac{A}{2}e^{j[\frac{2k_p\pi}{N}-\pi(\mathrm{N}-1)\frac{\delta}{N}-\varphi_0+\frac{\pi}{N}]}\cdot\frac{\sin(\pi\cdot\delta)}{\sin[\frac{\pi\cdot(2k_p+\delta+1)}{N}]} + W[k_p+1] \end{aligned} \quad (6)$$

Here, we define:

$$\mathrm{M} = \frac{R[k_p] - e^{-j\frac{\pi}{N}}\cdot\frac{\sin[\frac{\pi\cdot(2k_p+\delta+1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\delta)}{N}]}\cdot R[k_p+1]}{R[k_p] - e^{j\frac{\pi}{N}}\cdot\frac{\sin[\frac{\pi\cdot(2k_p+\delta-1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\delta)}{N}]}\cdot R[k_p-1]} \quad (7)$$

TABLE I
FREQUENCY ESTIMATION PROCEDURE

1	N-point DFT, search the peak sample from the first N/2 samples and its immediate left and right neighbors: $R[k_p]$, $R[k_p-1]$ and $R[k_p+1]$;
2	Set $\hat{\delta}^0=0$, For q=0 to Q-1 Calculate $\mathrm{M} = \frac{R[k_p] - e^{-j\frac{\pi}{N}}\cdot\frac{\sin[\frac{\pi\cdot(2k_p+\hat{\delta}^q+1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\hat{\delta}^q)}{N}]}\cdot R[k_p+1]}{R[k_p] - e^{j\frac{\pi}{N}}\cdot\frac{\sin[\frac{\pi\cdot(2k_p+\hat{\delta}^q-1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\hat{\delta}^q)}{N}]}\cdot R[k_p-1]}$ Obtain $\hat{\delta}^{q+1} = \frac{N}{\pi}\cdot\arctan(\tan\frac{\pi}{N}\cdot\frac{M-1}{M+1})$, Loop until q=Q-1
3	Obtain the estimated frequency $\hat{f}_0 = (k_p+\hat{\delta}^Q)\cdot f_s/N$

The definition of the parameter M is the key point in our method, which can help us get the closed-form expression of δ.

To derive our proposed estimator, let us only consider noise free case. The estimator's performance with respect to noise will be investigated later. Substituting Eq. (4), (5), (6) into (7), we obtain:

$$
\begin{aligned}
\mathrm{M} &= \frac{\frac{A}{2}e^{j[\varphi_0+\pi(\mathrm{N}-1)\frac{\delta}{N}]}\cdot\left\{\frac{\sin(\pi\cdot\delta)}{\sin(\frac{\pi\cdot\delta}{N})}-\frac{\sin(\pi\cdot\delta)}{\sin[\frac{\pi\cdot(\delta-1)}{N}]}\cdot\frac{\sin[\frac{\pi\cdot(2k_p+\delta+1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\delta)}{N}]}\right\}}{\frac{A}{2}e^{j[\varphi_0+\pi(\mathrm{N}-1)\frac{\delta}{N}]}\cdot\left\{\frac{\sin(\pi\cdot\delta)}{\sin(\frac{\pi\cdot\delta}{N})}-\frac{\sin(\pi\cdot\delta)}{\sin[\frac{\pi\cdot(\delta+1)}{N}]}\cdot\frac{\sin[\frac{\pi\cdot(2k_p+\delta-1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\delta)}{N}]}\right\}} \\
&= \frac{\sin[\frac{\pi\cdot(\delta+1)}{N}]}{\sin[\frac{\pi\cdot(\delta-1)}{N}]}\left\{\frac{\frac{\sin[\frac{\pi\cdot(\delta-1)}{N}]}{\sin(\frac{\pi\cdot\delta}{N})}-\frac{\sin[\frac{\pi\cdot(2k_p+\delta+1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\delta)}{N}]}}{\frac{\sin[\frac{\pi\cdot(\delta+1)}{N}]}{\sin(\frac{\pi\cdot\delta}{N})}-\frac{\sin[\frac{\pi\cdot(2k_p+\delta-1)}{N}]}{\sin[\frac{\pi\cdot(2k_p+\delta)}{N}]}}\right\} \\
&= \frac{\sin[\frac{\pi\cdot(\delta+1)}{N}]}{\sin[\frac{\pi\cdot(\delta-1)}{N}]}\left\{\frac{\left[\cos\frac{\pi}{N}-\sin\frac{\pi}{N}\cot(\frac{\pi\cdot\delta}{N})\right]-\left[\cos\frac{\pi}{N}+\sin\frac{\pi}{N}\cot\frac{\pi\cdot(2k_p+\delta)}{N}\right]}{\left[\cos\frac{\pi}{N}+\sin\frac{\pi}{N}\cot(\frac{\pi\cdot\delta}{N})\right]-\left[\cos\frac{\pi}{N}-\sin\frac{\pi}{N}\cot\frac{\pi\cdot(2k_p+\delta)}{N}\right]}\right\} \qquad (8) \\
&= -\frac{\sin[\frac{\pi\cdot(\delta+1)}{N}]}{\sin[\frac{\pi\cdot(\delta-1)}{N}]} = -\frac{\sin(\frac{\pi\cdot\delta}{N})\cos\frac{\pi}{N}+\cos(\frac{\pi\cdot\delta}{N})\sin\frac{\pi}{N}}{\sin(\frac{\pi\cdot\delta}{N})\cos\frac{\pi}{N}-\cos(\frac{\pi\cdot\delta}{N})\sin\frac{\pi}{N}} \\
&= \frac{\tan\frac{\pi}{N}+\tan(\frac{\pi\cdot\delta}{N})}{\tan\frac{\pi}{N}-\tan(\frac{\pi\cdot\delta}{N})}
\end{aligned}
$$

Then,

$$\tan(\frac{\pi\cdot\delta}{N}) = \tan\frac{\pi}{N}\cdot\frac{M-1}{M+1} \qquad (9)$$

$$\delta = \frac{N}{\pi}\cdot\arctan(\tan\frac{\pi}{N}\cdot\frac{M-1}{M+1}) \qquad (10)$$

According to Eq. (7) and Eq. (10), there are two unknowns (δ and M) in two equation, but unfortunately there is no closed-form solution, so we have to use numerical method to get the value of δ, which increases the computational complexity dramatically. In this paper, an iterative method is proposed to solve the problem and is described in Table I. With the proposed method, the value δ can be easily achieved using only three DFT samples, it also has a much lower computational complexity compared with the estimators utilizing a window function. The performance of the new method will be verified in next section.

IV. Performance Comparisons

This section presents the numerical comparison of the proposed estimator with the Orguner estimator [7], new three point estimator [8] and Borkowski estimator [11]. The Orguner

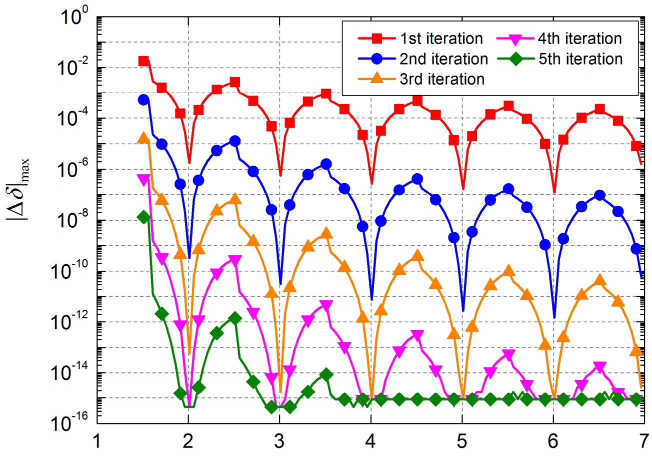

Fig. 2. Maximum magnitude of the frequency estimation error $|\Delta\delta|_{max}$ versus the sinusoid frequency v for different number of iteration. (N=32)

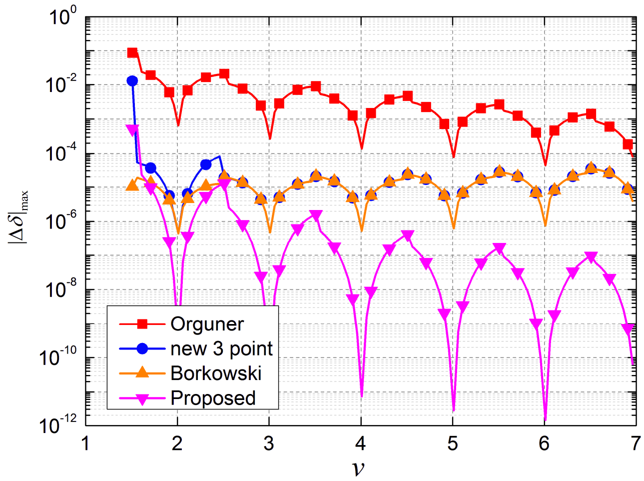

Fig. 3. Maximum magnitude of the frequency estimation error $|\Delta\delta|_{max}$ versus the sinusoid frequency v for different estimators. (N=32)

estimator is one of the best frequency estimators for complex signal. The new three point estimator based on the maximum sidelobe decay (MSD) window is highly effective in rejecting the detrimental effect on the estimation accuracy due to the negative frequency component. The Orguner estimator is also a very accurate estimator using MSD window which can eliminate the impact of negative spectrum component.

In the following comparison, all the estimators are based on three DFT samples, among which the Orguner estimator and the proposed estimator have lower complexity due to the absence of windowing.

First, the accuracy of the proposed method for different number of iteration is studied. The synthesized sine-wave amplitude is A = 1, the number of analyzed samples is N = 32. Fig. 2 shows the maximum frequency estimation error $|\Delta\delta|_{max}$ returned by simulations, as a function of the number of acquired sine-wave cycles v when pure sine-wave is considered. During simulations the value of v is varied in the range [1.51, 7) with a step of 0.05. For each value of v the maximum error was

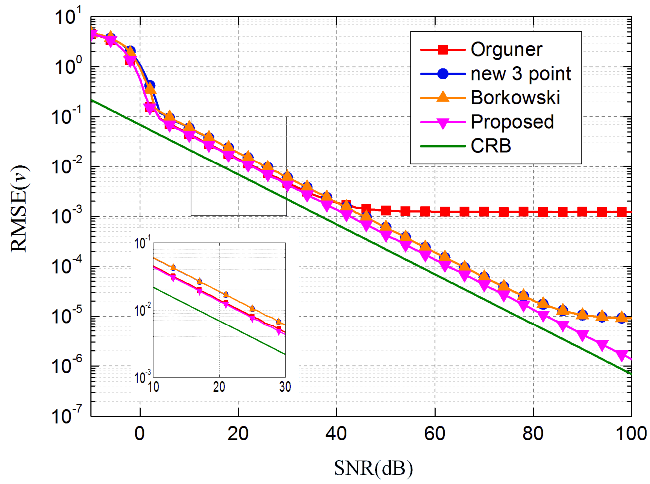

Fig. 4. Variation of RMSE with respect to SNR. (N=32)

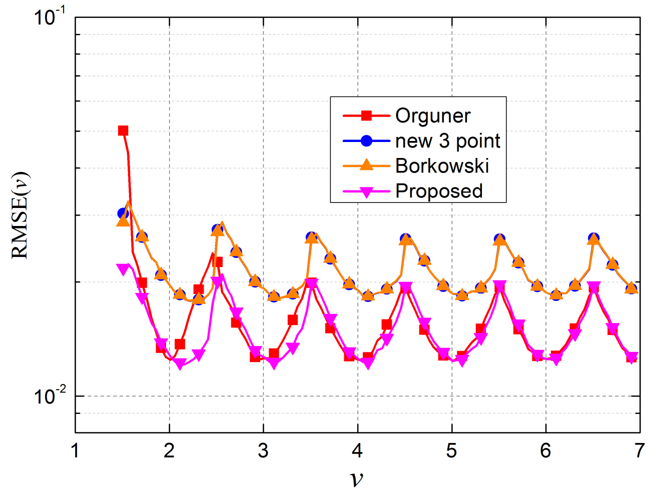

Fig. 5. Variation of RMSE with respect to the sinusoid frequency *v*. (SNR=20 dB, N=32)

determined by varying the sine-wave phase in the range [0, 2π) rad with a step of π/50 rad. As shown in Fig. 2, the frequency estimation error decreases as the number of iteration goes up. It indicate the iteration can improve the estimation accuracy.

Then, the accuracy performances of different frequency estimators are compared. For Orguner estimator and Borkowski estimator, the two-term MSD window (or Hann window) is adopted in simulations. For the proposed estimator, the error magnitudes are achieved after second iteration. Fig. 3 clearly confirms the proposed method outperforms the other competitors. If the number of iteration increases, the superiority of our proposed method will be more obvious.

Furthermore, the Root Mean Squared Error (RMSE) of the estimators and the Cramer–Rao lower bound are compared in the presence of Gaussian white noise. In Fig. 4, the normalized frequency *v* is fixed to be a specific value of 5.25(δ=0.25) and SNR is varied. As can be observed from this figure, the proposed estimator outperforms the other estimators at sufficiently high SNR values. That is because in high SNR range, the noiseless error dominates the RMSE, and the proposed method have the best accuracy. However, the RMSE difference is not so distinct in medium SNR range where the noise dominates the RMSE. We continue to investigate the performance of estimators in medium SNR range. In Fig. 5, SNR is fixed at 20 dB, it can be noted that the proposed estimator and Orguner estimator have slightly better results than the estimators with windowing technique. That is because the application of windowing results in the loss of output SNR [12].

V. Conclusion

In this paper, we have proposed a simple frequency estimator for real valued sinusoidal signals using only three DFT samples. In noiseless scenarios, the proposed estimator can achieve less estimation error compared with other estimators based on three DFT samples. In the case of Gaussian white noise, our proposed estimator outperforms the other alternatives in the high SNR range. For the medium SNR range, the proposed estimator also has slightly better performance compared with the estimators with window function.

References

[1] M. I. Skolnik, Introduction to Radar Systems. McGraw-Hill Book Company Inc, 1962.

[2] H. L. V. Trees, Detection, Estimation and Modulation Theory, Part1. New York: Wiley, 1971.

[3] B. G. Quinn, "Estimating frequency by interpolation using Fourier coefficients," IEEE Trans. Signal Processing, vol. 42, no. 5, pp. 1264–1268, May 1994.

[4] M. D. Macleod, "Fast nearly ML estimation of the parameters of real or complex single tones or resolved multiple tones," IEEE Trans. Signal Processing, vol. 46, no. 1, pp. 141–148, Jan. 1998.

[5] E. Jacobsen and P. Kootsookos, "Fast, accurate frequency estimators," IEEE Signal Process. Mag., vol. 24, pp. 123–125, May 2007.

[6] C. Candan, "A method for fine resolution frequency estimation from three DFT samples," IEEE Signal Process. Lett., vol. 18, no. 6, pp. 351–354, Jun. 2011.

[7] U. Orguner, C. Candan, "A fine-resolution frequency estimator using an arbitrary number of DFT coefficients," Signal Process., vol. 105, pp. 17–21, 2014.

[8] D. Belega, D. Petri and D. Dallet, "Frequency estimation of a sinusoidal signal via a three-point interpolated DFT method with high image component interference rejection capability," Digit. Signal Process., vol. 24, pp. 162–169, 2014.

[9] D. Belega and D. Petri, "Sine-wave parameter estimation by interpolated DFT method based on new cosine windows with high interference rejection capability," Digit. Signal Process., vol. 33, pp. 60–70, 2014

[10] C. Candan, "Fine resolution frequency estimation from three DFT samples: Case of windowed data," Signal Processing, vol. 114, no. 6, pp. 245–250, 2015.

[11] J. Borkowski, D. Kania, J. Mroczka, "Interpolated-DFT-Based Fast and Accurate Frequency Estimation for the Control of Power," IEEE Transactions on industrial Electronics, vol. 61, no. 12, pp. 7026–7034, 2014.

[12] B. Porat, A Course in Digital Signal Processing, New York: John Wiley & Sons, 1996.