The Range-Doppler SRL of Two Close Targets for Active Sensor in the GLRT Context

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***Abstract*—In this text we bring in the generalized likelihood radio test (GLRT) theory to resolve two-dimensional resolution problem of two close targets, naming the range-Doppler statistical resolution limit (SRL), which is put forwarded for the first time. We utilize the first-order Taylor expansion of the return signals to get an approximate linear model w.r.t. the range and Doppler parameters to be tested, then apply the detection theory to get a closed-form expression of the SRL. From the expression one can see the SRL is linear trade-off between the range and Doppler SRL, with their sum fixed. The two-dimension SRL is inversely proportional to the difference of their power. We have considered both cases with noise variance known or unknown. The numerical simulations have demonstrated the validity of our theoretical results.**

Keywords—GLRT; range-Doppler SRL; Taylor expansion; distinguishing rate

I. INTRODUCTION

The resolution limit for two close signals has drawn many researchers' interests in radar, sonar and other areas. Traditionally the Ambiguity Function(AF) [1] is a key tool to resolve it, whose result is known as the Rayleigh limit. However, the Rayleigh limit cannot reflect the influence of the ambient noise and/or clutter. Recently the decision theory has been applied to this problem, which regards the distinguishing process as a stochastic one, then the statistical resolution limit (SRL) is got. The SRL is the minimum distance between two targets along some dimension with a given distinguishing rate and a false-alarm rate, which has thrown a new sight onto this area and become a hot topic [2-6].

Among those existing literatures, some resolve this problem in range dimension[2,3] after match filtering, while others in one-dimensional angular [4] and two-dimensional angular [5], then further extend it to Doppler domain [6]. However, as far as we know, there is no similar work along range dimension for the signal before match filtering. As we all known, the match filtering can resort to high range sidelobe, which will make a difficulty in distinguishing the near targets. To avoid this problem and inspired by the former work [2-6], we will investigate the SRL along the range- and Doppler- dimension, which is denoted as range-Doppler SRL. We will utilize the tool of GLRT and get a close-formed expression of the SRL. Specifically, we will try to resolve that: whether there is one target or two targets in the cell of interest? How well the SRL can attain and what is the relations between the SRL with other parameters, such as SNR and the waveforms, etc. In fact, this is actually a counterpart problem with AF, which is also defined along the range- and Doppler- dimension. However, readers will find out that our conclusions within the GLRT framework are of great difference with that in AF.

The range-Doppler SRL is a multiple-dimension problem. In multiple-dimension SRL definition/derivation work, Korso has studied the multiple-variable SRL (MSRL) in [7-9], which is defined as the sum of the multiple absolute of the single SRL, expressed as $\delta = \sum_{p=1}^{P} |\delta_p|$, where p is the dimension number, and the relations of hypotheses-based MSRL to CRB-based MSRL has been presented. But their works are based on the assumption that these parameters should have the same unit[1], which cannot be utilized directly for our range-Doppler SRL. In fact, according to the definition of multiple parameter test [5,6] within the GLRT framework, the multiple variables can be modeled as a vector with all elements not being zero under the H_1 test, so the problem of units will not exist at all, as the test can collect the different items with normalized sum, which can be found in our later conclusions.

The rest of this paper is arranged as follows. Section II will present the test model, then an analytic expression of range-Doppler-SRL is given in Section III, based on which we will discuss how the corresponding parameters make effects on the value of the SRL. Finally some simulations will be presented in Section IV and Section V will make the conclusions.

Notations: Matrices and vectors are denoted by the bold capital letters and bold italic lowercase letters, respectively. Superscript $(\cdot)^T$, $(\cdot)^c$ and $(\cdot)^H$ denote transpose, conjugation and conjugate transpose of them, respectively. $\odot$ denotes the Hadamard product. $\Re(\bullet)$ denotes the real part of a complex number, while $\Im(\bullet)$ is the imaginary part of a complex

[1] Literation [7] has mentioned the multiple dimensions case with different units in an example, but this is not explicitly resolved.

number. $\mathbf{I}_N$ stands for a $N\times N$ identify matrix. $\dot{\mathbf{a}}$ denotes the differential of $\mathbf{a}$ to a parameter, in this paper we are indicating the time. $\lfloor \alpha \rfloor$ represents the nearest integer which is less than real number α.

II. Problem Formulation

For the active sensor, the return from the reflection of two close targets can be modeled as

$$x(t) = a_1 s(t-t_1)\mathrm{e}^{jw_{D_1}(t-t_1)} + a_2 s(t-t_2)\mathrm{e}^{jw_{D_2}(t-t_2)} + w'(t) \quad (1)$$

where a_1 and a_2 are the real magnitudes of the two signals, while $\{w_{D_i}\}_{i=1}^{2}$ and $\{t_i\}_{i=1}^{2}$ are the Doppler shift and delay time of them, respectively. As they are all unknown to the receiver, to make simplify and similar with the pioneering research work [2-6], we assume the central values of the Doppler shift and delay time are all known, denoting as $w_{D_0} = (w_{D_1} + w_{D_2})/2$ and $t_0 = (t_1 + t_2)/2$, respectively. $s(t)$ is the transmitted signal, which is known in the receiver in advanced, $w'(t)$ is complex white circularly Gaussian (CWCG) distribution, with mean zero and covariance σ^2.

Without loss of generality, we assume that $w_{D_1} < w_{D_2}, \mathrm{t}_1 < \mathrm{t}_2$ [2], make $\delta_w = w_{D_2} - w_{D_1}$ and $\delta_\mathrm{t} = t_2 - t_1$ denote the separations between the two signals. In order to remove the effect of them, we make a replacement with $l = t - t_0$, and multiply the term $e^{-jw_{D_0}l}$ to the two sides of (1), then apply the Taylor expansion with respective to the time l [3]. Finally we get the linear model $z(l) = x(l+t_0)e^{-jw_{D_0}l}$ to these parameters δ_t and δ_w, as

$$\begin{aligned} z(l) &= a_1 s(l+\frac{\delta_t}{2})\mathrm{e}^{j(w_{D_0}-\frac{\delta_w}{2})(l+\frac{\delta_t}{2})}\mathrm{e}^{-jw_{D_0}l} \\ &\quad + a_2 s(l-\frac{\delta_t}{2})\mathrm{e}^{j(w_{D_0}+\frac{\delta_w}{2})(l-\frac{\delta_t}{2})}\mathrm{e}^{-jw_{D_0}l} + w'(l+t_0)e^{-jw_{D_0}l} \\ &\approx a_1\left(s(l)+\frac{\delta_t}{2}\dot{s}(l)\right)\left(1+jw_{D_0}\frac{\delta_t}{2} - jl\frac{\delta_w}{2}\right) \\ &\quad + a_2\left(s(l)-\frac{\delta_t}{2}\dot{s}(l)\right)\left(1-jw_{D_0}\frac{\delta_t}{2} + jl\frac{\delta_w}{2}\right) + w(l) \\ &\approx (a_1+a_2)s(l) + \frac{\dot{s}(l)+js(l)w_{D_0}}{2}(a_1-a_2)\delta_t \\ &\quad - \frac{jls(l)}{2}(a_1-a_2)\delta_w + w(l) \end{aligned}, \quad (2)$$

where $w(l) = w'(l+t_0)e^{-jw_{D_0}l}$. For the first approximation we apply that $e^x \approx 1+x,\ x \ll 1$, which can only hold under the following conditions

$$w_{D_0}\frac{\delta_t}{2} \ll 1,\ l\frac{\delta_w}{2} \ll 1 \quad (3)$$

For the second approximation we drop the high-order and cross terms of δ_t and δ_w, as they are small comparing the reserving items .

Sampling the returning signals in (2), and use a vector expression as

$$\mathbf{x} = \left[x(-\frac{N-1}{2}), x(-\frac{N-1}{2}+1), \cdots, x(\frac{N-1}{2})\right]^T \quad (4)$$

for the $\mathbf{z}, \mathbf{s}, \mathbf{w}$, where $N = \lfloor \tau f_s \rfloor = \lfloor \tau / T_s \rfloor$ is the sampling number, f_s , T_s are the sampling frequency and period, respectively. To make the latter derivation simple, here we set the time index symmetric as $\sum_n (nT_s)^p f(nT_s) = 0$, where p is an odd number and $f(t)$ is even function.

With the vector expression in (4), the model in (2) can be rewritten as

$$\begin{aligned} \mathbf{z} &= \begin{bmatrix} (a_1+a_2)\mathbf{s} + \dfrac{\dot{\mathbf{s}} + jw_{D_0}\mathbf{s}}{2}(a_1-a_2)\delta_t \\ -\dfrac{j(T_s\mathbf{n})\odot\mathbf{s}}{2}(a_1-a_2)\delta_w + \mathbf{w} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \mathbf{s} & \dfrac{\dot{\mathbf{s}} + jw_{D_0}\mathbf{s}}{2} & -\dfrac{j(T_s\mathbf{n})\odot\mathbf{s}}{2} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} a_1+a_2 \\ (a_1-a_2)\delta_t \\ (a_1-a_2)\delta_w \end{bmatrix}}_{\boldsymbol{\theta}} + \mathbf{w}. \end{aligned} \quad (5)$$

where the $\mathbf{n} = -(N-1)/2 : (N-1)/2$ is the time sampling index, and the $\mathbf{n}T_s$ is the discrete time.

With the expression in (5), we can resolve the resolution problem as a hypothesis problem, where the H_0 denotes one signal present, and H_1 denotes two signals present. So the test model can be written as

$$\begin{cases} \mathrm{H}_0: & \mathbf{z} = \mathbf{h}_1\theta_1 + \mathbf{w} \\ \mathrm{H}_1: & \mathbf{z} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w} \end{cases}, \quad (6)$$

where $\mathbf{h}_1 = \mathbf{s}$, $\theta_1 = a_1 + a_2$, $\mathbf{H}$ and $\boldsymbol{\theta}$ are defined in (5). This is a classical linear model with the unknown parameters, which is equal to that $\delta_w = \delta_t = 0$ under H_0, and $\delta_w \neq 0$ or $\delta_t \neq 0$ under H_1, which is equal to $[\delta_w\ \ \delta_t]^T \neq \mathbf{0}$. In the next section we will discuss how to get an analysis expression of likelihood rate of the model (6) with the given signal $\mathbf{s}$.

III. The Derivation and Discussion of SRL

To get an analysis expression of the SRL, here we assume the signal is the Linear Frequency Modulation (LFM) pulse, and we will discuss two cases with the noise variance known or unknown.

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[2] There are other three cases, i.e., $w_{D_1} > w_{D_2}, \mathrm{t}_1 > \mathrm{t}_2$, $w_{D_1} > w_{D_2}, \mathrm{t}_1 < \mathrm{t}_2$ and $w_{D_1} < w_{D_2}, \mathrm{t}_1 > \mathrm{t}_2$, which will resort to the same results, so we take one as example.

[3] The variable meets the relation that $\delta_t B < 1$, otherwise there is pointless as we can distinguish them by matching filtering.

A. The Known Noise Variance Case

With the noise variance known, the model in (6) can be rewritten as

$$\begin{cases} H_0 : \mathbf{A\theta} = \mathbf{0}, a_1, a_2 \\ H_1 : \mathbf{A\theta} \neq \mathbf{0}, a_1, a_2 \end{cases}, \quad (7)$$

where $\boldsymbol{\theta} = [a_1 + a_2 \ (a_1 - a_2)\delta_t \ (a_1 - a_2)\delta_w]^T$. The unknown parameter are a_1 and a_2, the tested parameters are δ_t and δ_w. The selecting matrix $\mathbf{A}$ is expressed as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

As there are unknown parameters, so this is the composite test model, which can be solved by the GLRT theory. We first get the constrained most likelihood estimator (CMLE) of the unknown parameters $\hat{\boldsymbol{\theta}}_1 = (\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\Re\left(\mathbf{H}^{\mathrm{H}}\mathbf{z}\right)$ under H_1, then we get the GLRT statistic as

$$2\ln L_G(\mathbf{z}) = \frac{(\mathbf{A}\hat{\boldsymbol{\theta}}_1)^H[\mathbf{A}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{A}^T]^{-1}(\mathbf{A}\hat{\boldsymbol{\theta}}_1)}{\sigma^2/2} \gtrless \gamma_k. \quad (9)$$

Where γ_k is the detection threshold, and the subscript "*k*" stands for σ^2 is known. With the knowledge of the statistical signal processing, the performance of the test can be got by

$$2\ln L_G(\mathbf{z}) \overset{a}{\sim} \begin{cases} \chi_2^2, & H_0 \\ \chi_2'^2(\lambda_k(p_d, p_f)), & H_1 \end{cases},$$

where the non-centrality parameter is presented by

$$\lambda_k(p_d, p_f) = \frac{2\pi^2 K_D |a_1 - a_2|^2}{N^2\sigma^2} \times \left(\frac{\delta_w}{\delta_{w,rayleigh}} + \frac{\delta_t}{\delta_{t,rayleigh}}\right)^2, \quad (10)$$

where K_D is the square sum of time sampling number as

$$K_D = \sum_{k=-(N-1)/2}^{(N-1)/2} k^2 = \frac{(N^2-1)N}{12}. \quad (11)$$

The proof is presented in Appendix.

Readers should notice that, $\lambda_k(p_d, p_f)$ is key in illuminating the relationship between SRL and the other parameters. For simplicity, we remove the denotation of p_d and p_f of the parameter in the following text. Given $p_f = \alpha$, then the threshold and the distinguishing rate can be got as

$$\gamma_k = Q^{-1}_{\chi_p^2}(p_f), \ \ p_d = Q_{\chi_p'^2(\lambda_k)}(\gamma_k), \quad (12)$$

where $Q_{\chi_p^2}{}^{-1}(\cdot)$ is the inverse function of $Q_{\chi_p^2}(\cdot)$, which is the right-tail function of central Chi-Square distribution with *p* degree of freedoms(DOFs). $Q_{\chi_p'^2(\lambda_k)}(\cdot)$ is that of the noncentrality Chi-Square distribution with *p* DOFs, correspondingly its inverse function is $Q^{-1}_{\chi_p'^2(\lambda_k)}(\cdot)$.

B. The Unknown Noise Variance Case

When the noise variance is unknown, the model in (5) can be written as

$$\begin{cases} H_0 : \mathbf{A\theta} = \mathbf{0}, a_1, a_2, \sigma^2 > 0 \\ H_1 : \mathbf{A\theta} \neq \mathbf{0}, a_1, a_2, \sigma^2 > 0 \end{cases}, \quad (13)$$

where $\mathbf{A}$ and $\boldsymbol{\theta}$ are same with those in (6), but there is an additional unknown variable σ^2. As the variance is unknown, so we have to estimate it as well as the unknown magnitudes.

This model is related to a GLRT statistic as

$$2\ln L_G(\mathbf{z}) = (N-3)\frac{(\mathbf{A}\hat{\boldsymbol{\theta}}_1)^T[\mathbf{A}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{A}^T]^{-1}(\mathbf{A}\hat{\boldsymbol{\theta}}_1)}{\mathbf{z}^H(\mathbf{I} - \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H)\mathbf{z}} \gtrless \gamma_u, \quad (14)$$

where γ_u is the threshold which can be resolved similarly in the (12). where subscript "*u*" stands for σ^2 is unknown.

The performance of the statistic in (14) can be got by

$$2\ln L_G(\mathbf{z}) \overset{a}{\sim} \begin{cases} F_{2,N-3}, & H_0 \\ F'_{2,N-3}(\lambda_u), & H_1 \end{cases}, \quad (15)$$

where the noncentrality parameter is

$$\lambda_u = \frac{2\pi^2 K_D |a_1 - a_2|^2}{N^2\sigma^2}\left(\frac{\delta_w}{\delta_{w,rayleigh}} + \frac{\delta_t}{\delta_{t,rayleigh}}\right)^2, \quad (16)$$

which is same with (10), but one should note under this case the λ_u is related to a F distribution. The proof is also similar with the σ^2 known case, so we omit it here.

According to [11], the statistic is the central *F* distributed under H_0, and non-central *F* distributed under H_1, both with DOF pairs (2,*N*-3), as there are $rank\left(\mathbf{A}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{A}^T\right) = 2$ and $rank\left(\mathbf{I} - \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\right) = N-3$. Set $p_f = \alpha$, the threshold and the distinguishing rate can be got by

$$\gamma_u = Q^{-1}_{F_{2,N-3}}(p_f), p_d = Q_{F'_{2,N-3}(\lambda_u)}\left(\gamma_u\right), \quad (17)$$

where $Q_{F_{2,N-3}}(\cdot)$ is the right-tail function of central F distribution with DOF's pair ($2, N-3$), and $Q_{F'_{2,N-3(\lambda)}}(\cdot)$ is the right-tail function of non-centrality F distribution, with same DOFs, correspondingly their inverse functions are $Q^{-1}_{F_{T,N-3}}(\cdot)$ and $Q^{-1}_{F'_{2,N-3}(\lambda)}(\cdot)$, respectively.

C. The Influence Parameters on SRL

We denote the two-dimensional SRL as

$$\delta = \frac{\delta_w}{\delta_{w,rayleigh}} + \frac{\delta_t}{\delta_{t,rayleigh}}, \quad (18)$$

and define $\mathrm{SNR_D}$ as

$$\mathrm{SNR_D} = \frac{|a_1 - a_2|^2 \mathbf{s}^H\mathbf{s}}{\sigma^2} = \frac{|a_1 - a_2|^2 N}{\sigma^2},$$

then inserting K_D in the (11), the two-dimensional SRL in

(10) and (16) can be unified as

$$\delta = \frac{N\sigma}{\pi\left|a_1 - a_2\right|}\sqrt{\frac{\lambda}{2K_D}} \approx \frac{1}{\pi}\sqrt{\frac{6\lambda}{\mathrm{SNR_D}}}, \tag{19}$$

where the last approximation holds as the sampling number goes to infinity.

From the above expression, we can draw:

Remark 1: The two-dimensional SRL is the sum of normalized range-SRL and Doppler-SRL, which are normalized by the Rayleigh limit, $\delta_{w,rayleigh} = 2\pi/\tau$ and $\delta_{t,rayleigh} = 1/B$ respectively, so there are linear trade-off when their sum is fixed.

Remark 2: The two-dimensional SRL is inversely proportional to $\mathrm{SNR_D}$, which will become to be zero when the two signals are equal-powered, meaning that there is no resolvability in this case. This is of great difference with matching filtering (MF), whose result is proportional to the sum of signals' power. In fact they are still distinguished when equal-powered, if we adopt the high-order items of Taylor expansion in the approximation in (2). However, the distinguishing ability will become very poor as the high-order item is very small.

Remark 3: The two-dimensional SRL is relative to p_d and p_f through the noncentrality parameter λ. Specifically, a larger value of λ is needed in order to get a larger p_d with a fix p_f, or a smaller p_f with a fix p_d.

IV. Simulation Results

In this Section we will make several experiments to investigate our conclusions aforementioned. The LFM signal is expressed as $s(t) = e^{\mathrm{j}\pi Kt^2}$, $-\tau/2 \le t \le \tau/2$, which is at the zero carrier frequency, as the carrier frequency will not affect the results. The frequency modulation rate $K = B/\tau$, where the signal bandwidth $B = 1\mathrm{MHz}$ and time duration $\tau = 2\mu\mathrm{s}$. The sampling frequency is $f_s = 10B = 10\mathrm{MHz}$, and the center value of Doppler frequency $w_{D_0} = 1\mathrm{MHz}$. In all the experiments we set $p_f = 0.01$ to get the detection threshold, and the SNR is defined as $\left\{a_i^2/\sigma^2\right\}_{i=1,2}$.

In the first experiment we test our conclusions in both cases with noise variance known/unknown. First we get the specific threshold by the distribution under H_0 test with the pre-set p_f, see (12) and (17), then we get the corresponding p_d in two aspects. On one hand, we utilize the distribution function under H_1 hypothesis to get the theoretical value, also see (12) and (17). On the other hand, we make 10000 Monte Carlo simulations to generate the signal returns, then get the numerical value by get the statistical probability exceeding the threshold.

Fig. 1 plots the distinguishing rate versus SNR, where one can see that the numerical results coincide well with theoretical ones in both cases, with noise variance known and unknown. As expected, the known case has a better performance comparing to the unknown case. It can also be seen that the performance is poorest when the SNR1=SNR2, which is accord with the expression in (19) as the $\mathrm{SNR_D}$ is in the denominator.

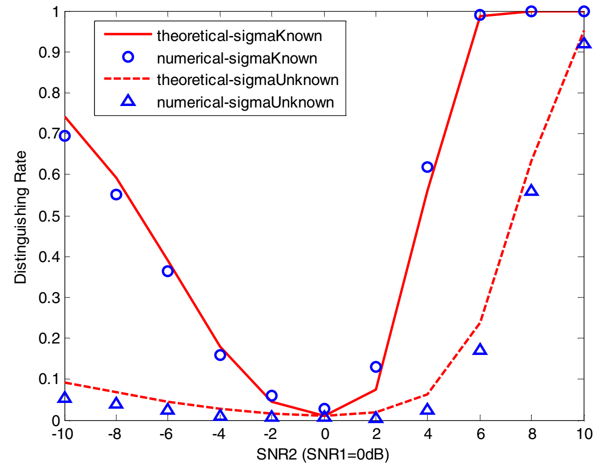

Fig.1 Distinguishing rate versus SNR with the normalized limits $\delta_t/\delta_{t,rayleigh} = 0.2$ and $\delta_w/\delta_{w,rayleigh} = 0.2$

The second example is to test the distinguishing rate versus the signal bandwidth B and duration time τ with the noise variance known, which is plotted in Fig.2. We set the separation. $\delta_t = 0.2\mu\mathrm{s}$, $\delta_w = 0.2M\mathrm{Hz}$ and $\mathrm{SNR}_D = 0dB$, and vary the B and τ. It can be seen that the performance improves as either the value of B or τ increases. This is obvious from (18), which shows that the two dimensional SRL is the sum of the two, so the improvement of either will be constructive to the results.

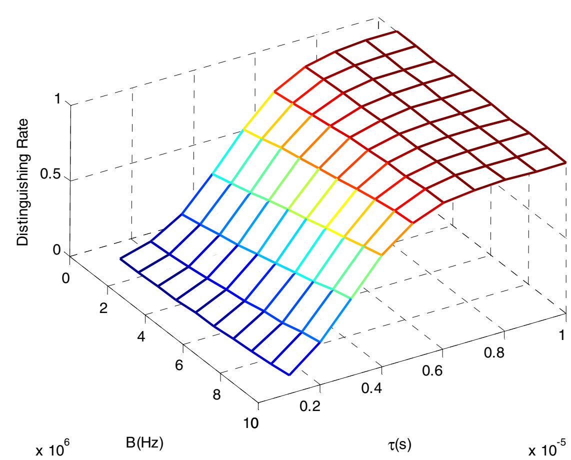

Fig.2 Distinguishing rate versus B and τ, with $\mathrm{SNR}_D = 0dB$ and $\delta_t = 0.2\mu\mathrm{s}$ and $\delta_w = 0.2M\mathrm{Hz}$

To dwell on the relations of two distinguishing parameters, we plot Fig.3 in terms of different non-centrality value λ, which is related a specified value of distinguishing rate, under the given p_f and $\mathrm{SRN_D}$. We get a group of lines, indicating that the range-SRL and Doppler-SRL are linear trade-off with their sum fixed. It can also be exploited that a large λ is corresponding to a large p_d, which is accord with our remark in the last Section.

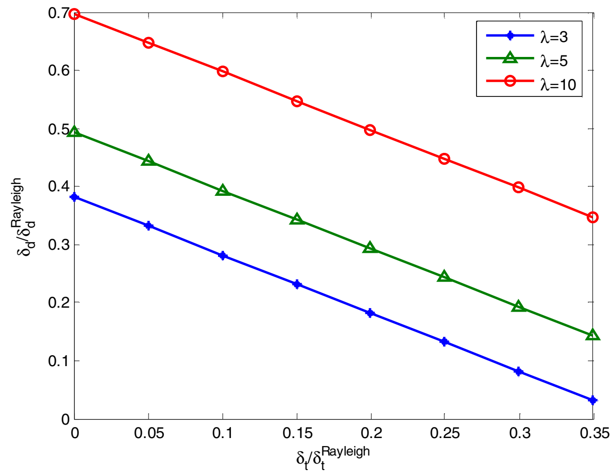

Fig. 3 Trade off between the two SRLs with the given value of non-centrality parameter, which is related to a set of P_d and P_f

V. CONCLUSIONS

In this paper, the range-Doppler SRL for two near targets is investigated in the GLRT framework. We get a closed-form expression after exploiting Taylor expansion and first-order approximation to the original signals. Based on the expression. One can see that the two-dimensional SRL is the sum of the two single dimension SRLs, which is inversely proportional to the difference powers of the two signals with a factor of 1/2. In fact, as the Doppler shift is tiny for the single pulse, we can explore this problem in multiple pulses. To make the conclusions more accurate, more items reserving is recommended for the Taylor expansion.

APPENDIX DERIVATION OF (8)

As the distribution of the return signal in (5) is complex Gaussian under both hypotheses, the negative log-likelihood function under H_1 is given by

$$L(\mathbf{z},\boldsymbol{\theta}) = -\ln p(\mathbf{y}) = N\ln(\pi\sigma^2)+\sigma^{-2}\left\|\mathbf{z}-\mathbf{H}\boldsymbol{\theta}\right\|^2 \qquad (20)$$

As all the parameters to be estimated are all real numbers, this is a problem of CMLE, we can utilize the Lagrange multiplier method to get the optimum value.

The constrained condition is $\Im(\boldsymbol{\theta})=0$, which is equal to $j(\boldsymbol{\theta}-\boldsymbol{\theta}^c)=0$. Let μ be a real Lagrange multiplier, then the Lagrange function is given by

$$\Pi(\mathbf{y},\boldsymbol{\theta})=L(\mathbf{y},\boldsymbol{\theta})+\mu j(\boldsymbol{\theta}-\boldsymbol{\theta}^c) \qquad (21)$$

Then, the partial derivatives of the Lagrange function are

$$\begin{cases}\dfrac{\partial\Pi}{\partial\boldsymbol{\theta}}=\dfrac{1}{\sigma^2}\left(\mathbf{H}^H\mathbf{H}\boldsymbol{\theta}^c-\mathbf{H}^H\boldsymbol{\theta}\right)+\mu j=0\\ \dfrac{\partial\Pi}{\partial\mu}=j\left(\boldsymbol{\theta}-\boldsymbol{\theta}^c\right)=0\end{cases} \qquad (22)$$

Resolving the (22), we get the constrained MLE of the parameters to be estimated as

$$\hat{\boldsymbol{\theta}}_1=\frac{\Re(\mathbf{H}^{\mathrm{H}}\mathbf{y})}{\mathbf{H}^{\mathrm{H}}\mathbf{H}} \qquad (23)$$

As the noise is belonging to the CWCG distribution, so there are $\Re\{\mathbf{w}\}\sim\mathcal{N}\left(0,\left(\sigma^2/2\right)\mathbf{I}\right)$ and $\Im\{\mathbf{w}\}\sim\mathcal{N}\left(0,\left(\sigma^2/2\right)\mathbf{I}\right)$, and then we can get the covariance of the new variable $\mathbf{t}=\Re(\mathbf{H}^{\mathrm{H}}\mathbf{y})$ (its mean is $\mathbf{H}^{\mathrm{H}}\mathbf{H}\boldsymbol{\theta}$) as

$$\begin{aligned}C_{\mathbf{t}} &= E\left\{\left(\Re\left(\mathbf{H}^{\mathrm{H}}\mathbf{y}\right)-\mathbf{H}^{\mathrm{H}}\mathbf{H}\boldsymbol{\theta}\right)^2\right\}=E\left\{\left(\Re\left(\mathbf{H}^{\mathrm{H}}\mathbf{w}\right)\right)^2\right\}\\ &= E\left\{\left(\Re\left(\mathbf{H}^{\mathrm{H}}\right)\Re(\mathbf{w})-\Im\left(\mathbf{H}^{\mathrm{H}}\right)\Im(\mathbf{w})\right)^2\right\}\\ &= E\left\{\left(\Re(\mathbf{H})^T\Re(\mathbf{w})+\Im(\mathbf{H})^T\Im(\mathbf{w})\right)^2\right\}\end{aligned} \qquad (24)$$

Using the circularity of the noise, which meets that

$$E\left(\Re(\mathbf{w})\Im(\mathbf{w})^T\right)=E\left(\Im(\mathbf{w})\Re(\mathbf{w})^T\right)=\mathbf{0} \qquad (25)$$

and

$$E\left(\Re(\mathbf{w})\Re(\mathbf{w})^T\right)=E\left(\Im(\mathbf{w})\Im(\mathbf{w})^T\right)=\frac{\sigma^2}{2}\mathbf{I}, \qquad (26)$$

so we get that

$$C_{\mathbf{t}}=\frac{\sigma^2}{2}\mathbf{H}^{\mathrm{H}}\mathbf{H}. \qquad (27)$$

According to [10, Appendix 7B], we can obtain a new statistic as

$$2\ln L_G(\mathbf{z})=\frac{(\mathbf{A}\hat{\boldsymbol{\theta}}_1)^H[\mathbf{A}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{A}^T]^{-1}(\mathbf{A}\hat{\boldsymbol{\theta}}_1)}{\sigma^2/2} \sim\begin{cases}\chi_2^2\\ \chi_2^{'2}(\lambda_k)\end{cases}, \qquad (28)$$

where A is presented in (8).

Denote $\boldsymbol{s}$ as the discrete sampling of a LFM signal, as

$$s(n)=e^{j\pi\frac{B}{\tau}n^2T_s^2},\quad -\frac{N-1}{2}\le n\le\frac{N-1}{2}, \qquad (29)$$

where $N=\lfloor\tau f_s\rfloor$, τ is the time duration and B is the signal bandwidth. f_s and T_s are the sampling frequency and period, respectively.

Inserting to the expression of $\mathbf{H}$ in (5), then we obtain

$$\mathbf{H}^H\mathbf{H}=\begin{bmatrix}N & j\dfrac{w_{D_0}N}{2} & 0\\ -j\dfrac{w_{D_0}N}{2} & \dfrac{\pi^2B^2K_D}{N^2}+\dfrac{w_{D_0}^2N}{4} & \dfrac{\pi BK_DT_s}{2N}\\ 0 & \dfrac{\pi BK_DT_s}{2N} & \dfrac{K_DT_s^2}{4}\end{bmatrix}, \qquad (30)$$

where K_D is given in (11).

According to the block matrix's reverse lemma, we can get that

$$[\mathbf{A}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{A}^T]^{-1} = \left(\begin{bmatrix} \frac{\pi^2 B^2 K_D}{N^2} + \frac{w_{D_0}^2 N}{4} & \frac{\pi B K_D T_s}{2N} \\ \frac{\pi B K_D T_s}{2N} & \frac{K_D T_s^2}{4} \end{bmatrix} - \frac{1}{N} \begin{bmatrix} -j\frac{w_{D_0} N}{2} \\ 0 \end{bmatrix} \begin{bmatrix} j\frac{w_{D_0} N}{2} & 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{\pi^2 B^2 K_D}{N^2} & \frac{\pi B K_D T_s}{2N} \\ \frac{\pi B K_D T_s}{2N} & \frac{K_D T_s^2}{4} \end{bmatrix} \quad (31)$$

And then DOF of the statistic is decided by $rank\left(\mathbf{A}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{A}^T\right) = 2$. Then the non-central parameter value can be got as

$$\lambda_k = \frac{(\mathbf{A}\boldsymbol{\theta}_1)^H \left(\mathbf{A}\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{A}^T\right)^{-1}(\mathbf{A}\boldsymbol{\theta}_1)}{\sigma^2/2} = \frac{2\pi^2 K_D (a_1 - a_2)^2}{N^2\sigma^2}\left(\frac{\delta_w}{\delta_{w,rayleigh}} + \frac{\delta_t}{\delta_{t,rayleigh}}\right)^2, \quad (32)$$

where $\boldsymbol{\theta}_1$ is the true value of test parameters, and the Rayleigh limits of the two parameters are $\delta_{w,rayleigh} = 2\pi/\tau$ and $\delta_{t,rayleigh} = 1/B$ respectively, so the expression of λ_k in (10) follows.

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